# THE CALCULATION OF THE ASTRONOMICAL REFRACTION FOR THE ELLIPSOIDAL ATMOSPHERE OF THE EARTH AT GREAT ZENITH DISTANCES

MINODORA CARMEN LIPCANU

*University of Bucharest, Faculty of Mathematics and Informatics*
*Str. Academiei 14, RO-70109 Bucharest, Romania*
*E-mail: mlipcanu@gmail.com*

*Abstract*. The astronomical refraction for the ellipsoidal atmosphere of the Earth is calculated, when the light ray situated in the meridian plane of the place is observed at a zenith distance included between 70° and 81° .The calculation is made using the model of the terrestrial atmosphere TSA-60. It represents an extension of the calculation of the astronomical refraction for the ellipsoidal atmosphere at zenithal distances below 70°, which was previously calculated (Mihăila and Lipcanu 2004). By comparing these results to the ones of the spherical model, it is ascertained that in this case also the refraction calculated for the ellipsoidal model is greater than the one for the spherical model, the difference increasing with the zenith distance and depending on the latitude of the place.

*Key words*: astrometry - astronomical refraction - atmosphere models.

## 1. INTRODUCTION

Even from the beginning of the XX$^{th}$ century, a model of the atmosphere of the Earth has been searched, which may allow the calculation of the refraction with a better approximation than in the case of the classical model of the atmosphere, constituted by concentric spherical layers. An initial model was proposed by Harzer (Harzer 1922, 1924), who took into consideration the Earth as being a revolution ellipsoid, and the atmosphere as being constituted of optic surfaces. This model was improved by Sergienko (Sergienko 1979), Shabelnikov (Shabelnikov 1983), Yatsenko and Teleki (Yatsenko and Teleki 1985; Yatsenko 1995), the calculation still being time-taking. A different model, geometrically simple, was

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provided by Mihăilă (Mihăilă 1973), who considered the atmosphere as an ellipsoidal layer.

In the following, making use of the refraction integral deduced by Mihăilă (Mihăilă 1973), we shall present the calculation of the refraction for the ellipsoidal model, for zenith distances between 70° and 81°, when the refracted ray is situated in the meridian plane of the observational place, and the obtained results will be compared to those from the spherical model. In a previous study this formula was applied for zenith distances below 70° (Mihăilă and Lipcanu 2004), and in this study the series of refraction will be extended with three more terms. For this purpose, the model of atmosphere TSA-60 will be applied (see, Dinulescu 1967), because in this case the refraction depends of the structure of the atmosphere. The model is given for the latitude of approximately 45°. The layers of equal density can be considered ellipsoidal layers. The method of Leone (Leone 1962, Dinulescu 1967) will be used, which considered that the TSA-60 model is spherical.

## 2. REFRACTION SERIES

It is assumed that the atmosphere of the Earth is constituted by ellipsoidal layers of constant density. The separation surfaces of the layers are homothetic ellipsoids, one of these being the ellipsoid of the Earth. The intersections of these surfaces with the meridian plane of the observational place will be homothetic concentric ellipses. For this model, Mihăilă (Mihăilă 1973) obtained the following formula for the integral of refraction

$$R_\varphi = \int_1^{n_0} \frac{r_0 n_0 \sin z}{\sqrt{r^2 n^2 - r_0^2 n_0^2 \sin^2 z}} \frac{dn}{n}, \tag{1}$$

where $z$ is the observed zenith distance, $r$ is the radius of curvature of the ellipse which passes through the current point $P$ on the light ray trajectory and $n$ the refraction index of the atmospheric layer in which the considered point is situated. The suffix zero signifies the data from the observational place $P_0$. The integral has the same form as in the case of the spherical model, with the simple difference that the radius of curvature comes instead of the geocentric distance. The relation (1) can also be written as

$$R_\varphi = \int_1^{n_0} \frac{\frac{r_0}{r}\frac{n_0}{n}\tan z}{\sqrt{1+\left(1-\frac{r_0^2}{r^2}\frac{n_0^2}{n^2}\right)\tan^2 z}} \frac{dn}{n}. \tag{2}$$

We shall introduce the following notation

$$2u_\varphi = 1 - \frac{n_0^2}{n^2}\frac{r_0^2}{r^2}. \tag{3}$$

For $z \le 81^\circ$, $2|u_\varphi|\tan^2 z$ takes lesser values than 1, and the following development can be used

$$(1 + 2u_\varphi \tan^2 z)^{-\frac{1}{2}} = 1 - u_\varphi \tan^2 z + \frac{3}{2}u_\varphi^2 \tan^4 z - \frac{5}{2}u_\varphi^3 \tan^6 z + \frac{35}{8}u_\varphi^4 \tan^8 z - \ldots \tag{4}$$

Performing the replacement in relation (2), it is obtained

$$R_\varphi = \tan z \int_1^{n_0} \frac{r_0}{r} d\left(\frac{n_0}{n}\right) - \tan^3 z \int_1^{n_0} \frac{r_0}{r} u_\varphi d\left(\frac{n_0}{n}\right) + \frac{3}{2}\tan^5 z \int_1^{n_0} \frac{r_0}{r} u_\varphi^2 d\left(\frac{n_0}{n}\right) -$$

$$- \frac{5}{2}\tan^7 z \int_1^{n_0} \frac{r_0}{r} u_\varphi^3 d\left(\frac{n_0}{n}\right) + \frac{35}{8}\tan^9 z \int_1^{n_0} \frac{r_0}{r} u_\varphi^4 d\left(\frac{n_0}{n}\right) - \ldots$$

From this series, only the first five terms have significance for z ≤ 81°, with a 0″.001 error. Keeping only these terms, we have

$$R_\varphi = A_{0\varphi} \tan z - A_{1\varphi} \tan^3 z + \frac{3}{2} A_{2\varphi} \tan^5 z - \frac{5}{2} A_{3\varphi} \tan^7 z + \frac{35}{8} A_{4\varphi} \tan^9 z, \tag{5}$$

where

$$A_{i\varphi} = \int_1^{n_0} \frac{r_0}{r} u_\varphi^i d\left(\frac{n_0}{n}\right), i = 0, \ldots, 4. \tag{6}$$

Next, the coefficients (6) will be determined following the procedure used by Leone (Leone 1962) in the case of the spherical model. For simplification purposes the notations utilized in the case of the spherical atmosphere will be used

$$N = \frac{n}{n_0}, Q = \frac{r}{r_0}, N' = \frac{1}{n_0}, Q' = \frac{r_{\max}}{r_0}. \tag{7}$$

The index „max" refers to the last ellipsoidal layer, significant for the refraction, situated at the altitude of 80 km. With the expressions (7), the coefficients (6) become

$$A_{i\varphi} = \int_{N'}^{1} \frac{u_\varphi^i}{QN^2} dN, u_\varphi = \frac{1}{2}\left(1 - \frac{1}{N^2 Q^2}\right). \tag{8}$$

Thus, the coefficients can be written recursively

$$
\begin{aligned}
A_{0\varphi} &= \int_{N'}^{1} \frac{1}{QN^2}\, dN, \\
A_{1\varphi} &= \frac{1}{2} A_{0\varphi} - \frac{1}{2} \int_{N'}^{1} \frac{1}{Q^3 N^4}\, dN, \\
A_{2\varphi} &= -\frac{1}{4} A_{0\varphi} + A_{1\varphi} + \frac{1}{4} \int_{N'}^{1} \frac{1}{Q^5 N^6}\, dN, \\
A_{3\varphi} &= \frac{1}{8} A_{0\varphi} - \frac{3}{4} A_{1\varphi} + \frac{3}{2} A_{2\varphi} - \frac{1}{8} \int_{N'}^{1} \frac{1}{Q^7 N^8}\, dN, \\
A_{4\varphi} &= -\frac{1}{16} A_{0\varphi} + \frac{1}{2} A_{1\varphi} - \frac{3}{2} A_{2\varphi} + 2A_{3\varphi} + \frac{1}{16} \int_{N'}^{1} \frac{1}{Q^9 N^{10}}\, dN.
\end{aligned}
\tag{9}
$$

We consider also

$$
\beta = \frac{n_0}{Q'}. \tag{10}
$$

After the integration by parts in relations (9), the refraction coefficients in formula (5) can be written as

$$
\begin{aligned}
A_{0\varphi} &= \beta - 1 + \int_{1}^{R'} \frac{dQ}{NQ^2}, \\
A_{1\varphi} &= \frac{1}{2} A_{0\varphi} - \frac{1}{6}\left(\beta^3 - 1\right) - \frac{1}{2} \int_{1}^{R'} \frac{dQ}{N^3 Q^4}, \\
A_{2\varphi} &= -\frac{1}{4} A_{0\varphi} + A_{1\varphi} + \frac{1}{20}\left(\beta^5 - 1\right) + \frac{1}{4} \int_{1}^{R'} \frac{dQ}{N^5 Q^6}, \\
A_{3\varphi} &= \frac{1}{8} A_{0\varphi} - \frac{3}{4} A_{1\varphi} + \frac{3}{2} A_{2\varphi} - \frac{1}{56}\left(\beta^7 - 1\right) - \frac{1}{8} \int_{1}^{R'} \frac{dQ}{N^7 Q^8}, \\
A_{4\varphi} &= -\frac{1}{16} A_{0\varphi} + \frac{1}{2} A_{1\varphi} - \frac{3}{2} A_{2\varphi} + 2A_{3\varphi} + \frac{1}{144}\left(\beta^9 - 1\right) + \frac{1}{16} \int_{1}^{R'} \frac{dQ}{N^9 Q^{10}}.
\end{aligned}
\tag{11}
$$

The Gladstone relation will give for the considered model

$$
\frac{n-1}{\delta} = \frac{n_0 - 1}{\delta_0} = \rho_0 = 0.2389388\,, \tag{12}
$$

where $\delta_0$=1.2250x10$^{-3}$ for TSA-60 and $n_0$=1.0002927 (Danjon 1959). From (7) and (12) it is obtained, if we limit to the terms of order 4 in $\delta$ and $\rho_0$,

$$\begin{aligned}
\frac{1}{N} &= n_0(1+\rho_0\delta)^{-1} = n_0\left(1-\rho_0\delta+\rho_0^2\delta^2-\rho_0^3\delta^3+\rho_0^4\delta^4\right),\\
\frac{1}{N^3} &= n_0^3(1+\rho_0\delta)^{-3} = n_0^3\left(1-3\rho_0\delta+6\rho_0^2\delta^2-10\rho_0^3\delta^3+15\rho_0^4\delta^4\right),\\
\frac{1}{N^5} &= n_0^5(1+\rho_0\delta)^{-5} = n_0^5\left(1-5\rho_0\delta+15\rho_0^2\delta^2-35\rho_0^3\delta^3+70\rho_0^4\delta^4\right),\\
\frac{1}{N^7} &= n_0^7(1+\rho_0\delta)^{-7} = n_0^7\left(1-7\rho_0\delta+28\rho_0^2\delta^2-84\rho_0^3\delta^3+210\rho_0^4\delta^4\right),\\
\frac{1}{N^9} &= n_0^9(1+\rho_0\delta)^{-9} = n_0^9\left(1-9\rho_0\delta+45\rho_0^2\delta^2-165\rho_0^3\delta^3+495\rho_0^4\delta^4\right).
\end{aligned} \tag{13}$$

Introducing these values in the integrals (11), the following results are obtained

$$\begin{aligned}
A_{0\varphi} &= n_0-1-\rho_0 n_0\int_1^{R'}\delta\frac{dQ}{Q^2}+\rho_0^2 n_0\int_1^{R'}\delta^2\frac{dQ}{Q^2}-\rho_0^3 n_0\int_1^{R'}\delta^3\frac{dQ}{Q^2}+\rho_0^4 n_0\int_1^{R'}\delta^4\frac{dQ}{Q^2},\\
A_{1\varphi} &= \frac{1}{2}A_{0\varphi}-\frac{1}{6}\left(n_0^3-1\right)+\frac{3}{2}\rho_0 n_0^3\int_1^{R'}\delta\frac{dQ}{Q^4}-3\rho_0^2 n_0^3\int_1^{R'}\delta^2\frac{dQ}{Q^4}+\\
&\quad+5\rho_0^3 n_0^3\int_1^{R'}\delta^3\frac{dQ}{Q^4}-\frac{15}{2}\rho_0^4 n_0^3\int_1^{R'}\delta^4\frac{dQ}{Q^4},\\
A_{2\varphi} &= -\frac{1}{4}A_{0\varphi}+A_{1\varphi}+\frac{1}{20}\left(n_0^5-1\right)-\frac{5}{4}\rho_0 n_0^5\int_1^{R'}\delta\frac{dQ}{Q^6}+\frac{15}{4}\rho_0^2 n_0^5\int_1^{R'}\delta^2\frac{dQ}{Q^6}-\\
&\quad-\frac{35}{4}\rho_0^3 n_0^5\int_1^{R'}\delta^3\frac{dQ}{Q^6}+\frac{35}{2}\rho_0^4 n_0^5\int_1^{R'}\delta^4\frac{dQ}{Q^6},\\
A_{3\varphi} &= \frac{1}{8}A_{0\varphi}-\frac{3}{4}A_{1\varphi}+\frac{3}{2}A_{2\varphi}-\frac{1}{56}\left(n_0^7-1\right)+\frac{7}{8}\rho_0 n_0^7\int_1^{R'}\delta\frac{dQ}{Q^8}-\frac{7}{2}\rho_0^2 n_0^7\int_1^{R'}\delta^2\frac{dQ}{Q^8}+\\
&\quad-\frac{21}{2}\rho_0^3 n_0^7\int_1^{R'}\delta^3\frac{dQ}{Q^8}-\frac{105}{4}\rho_0^4 n_0^7\int_1^{R'}\delta^4\frac{dQ}{Q^8},\\
A_{4\varphi} &= -\frac{1}{16}A_{0\varphi}+\frac{1}{2}A_{1\varphi}-\frac{3}{2}A_{2\varphi}+2A_{3\varphi}+\frac{1}{144}\left(n_0^9-1\right)-\frac{9}{16}\rho_0 n_0^9\int_1^{R'}\delta\frac{dQ}{Q^{10}}+\\
&\quad+\frac{45}{16}\rho_0^2 n_0^9\int_1^{R'}\delta^2\frac{dQ}{Q^{10}}-\frac{165}{16}\rho_0^3 n_0^9\int_1^{R'}\delta^3\frac{dQ}{Q^{10}}+\frac{495}{16}\rho_0^4 n_0^9\int_1^{R'}\delta^4\frac{dQ}{Q^{10}}.
\end{aligned} \tag{14}$$

For the calculation of the integrals from the expressions above, we considered a division of the atmosphere, of the TSA-60 model, constituted of 80 concentric ellipsoidal layers, having each the thickness of 1 km. For the density values, the arithmetic mean of the extreme values was taken at each layer. Obviously, the calculation of the coefficients will be more precise, the greater the number of the atmospheric layers is. The index $\varphi$ from the relations (14) appears also in the expression of $Q'$. Thus, using the expression of the radius of curvature of a point on the ellipse

$$r = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\varphi\right)^{3/2}}, \tag{15}$$

where $a$ is the semiaxe of the ellipse, $e$ its excentricity, $\varphi$ the latitude of the point, using (7) we obtain

$$Q' = \frac{a_{\max}}{a_0}\left(\frac{1-e^2\sin^2\varphi_0}{1-e^2\sin^2\varphi_{\max}}\right)^{\frac{3}{2}} = \frac{a_{\max}}{a_0}\left[\left(1-e^2\sin^2\varphi_0\right)\left(1-e^2\sin^2\varphi_{\max}\right)^{-1}\right]^{\frac{3}{2}}, \tag{16}$$

where $\varphi_{\max}$ is the geodetic latitude of point $P_{\max}$, where the radiation penetrates the terrestrial atmosphere at the superior limit of the atmosphere. This is obtained approximately, determining the position of the point situated at the intersection of the tangent to trajectory in $P_0$ with the ellipse corresponding to the superior limit of the atmosphere. Only the atmospheric layer of 80 km thickness presents sensitive effect for the refraction.

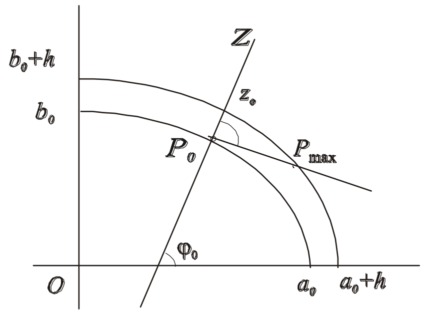


Fig. 1. – The configuration from the meridian plan of the place

The coordinates of the point of entering the atmosphere are given by the system

$$\begin{cases} y - y_0 = (x - x_0)\tan(\varphi_0 - z), x_0 = \dfrac{a_0 \cos\varphi_0}{\sqrt{1 - e^2 \sin^2 \varphi_0}}, y_0 = \dfrac{a_0 (1 - e^2)\sin\varphi_0}{\sqrt{1 - e^2 \sin^2 \varphi_0}}, \\ \left(\dfrac{x}{a_0 + h}\right)^2 + \left(\dfrac{y}{b_0 + h}\right)^2 = 1, \end{cases} \tag{17}$$

where $a_0$ and $b_0$ are the semiaxes of the terrestrial meridian.

If ($x_{max}$, $y_{max}$) is the solution of system (17), then $\varphi_{max} = \arctan \dfrac{y_{max}}{x_{max}} \left(\dfrac{a_0 + h}{b_0 + h}\right)^2$.

The product $e^2 \sin^2 \varphi_0$ is very small, between 0 and $6.4 \cdot 10^{-3}$. Therefore, developing in series and keeping the first two terms, the result is

$$Q' = \frac{a_{max}}{a_0}\left[1 + e^2\left(\sin^2 \varphi_{max} - \sin^2 \varphi_0\right)\right]^{\frac{3}{2}}. \tag{18}$$

Finally, it is obtained

$$Q' = \frac{a_{max}}{a_0}\left[1 + e^2 \sin(\varphi_{max} + \varphi_0)\sin(\varphi_{max} - \varphi_0)\right]^{\frac{3}{2}}. \tag{19}$$

Because the product $\left|e^2 \sin(\varphi_{max} - \varphi_0)\sin(\varphi_{max} + \varphi_0)\right| \le 6.4 \cdot 10^{-3}$, developing in series and retaining the first two terms, we can write

$$Q' = \frac{a_{max}}{a_0}\left[1 + \frac{3}{4}e^2(\cos 2\varphi_0 - \cos 2\varphi_{max})\right]. \tag{20}$$

This expression was used for the calculation of the coefficients (14).

## 3. RESULTS

In order to compare the results obtained in the case of the ellipsoidal model, using the formula (5), with the coefficients (14), the refraction in the case of the spherical model was calculated for the same zenithal distances between 70° and 81°. In this case the expression (5) becomes

$$R = A_0 \tan z - A_1 \tan^3 z + \frac{3}{2} A_2 \tan^5 z - \frac{5}{2} A_3 \tan^7 z + \frac{35}{8} A_4 \tan^9 z, \tag{21}$$

where the coefficients do not depend on $\varphi$, and the variable $Q$ becomes $Q = \frac{r}{r_0}$, where $r_0$ is the radius of the earth and $r$ is the radius of the spherical layer. In this case $Q' = \frac{r_{max}}{r_0}$, where $r_{max}$ is the radius of the layer with the altitude $h$= 80 km.

In table 1 we present the values of the refraction $R$ (in arcsec), calculated by formula (21), for each degree, for zenith distance $z$ between 70° and 81°.

*Table 1*
Refraction for spherical model

| $z$ | $R$ | $z$ | $R$ |
|---|---|---|---|
| 70 | 164.218 | 76 | 237.466 |
| 71 | 173.413 | 77 | 255.714 |
| 72 | 183.556 | 78 | 276.749 |
| 73 | 194.810 | 79 | 301.263 |
| 74 | 207.370 | 80 | 330.191 |
| 75 | 221.485 | 81 | 364.842 |

Table 2 presents the deviations $\Delta(z,\varphi) = R_\varphi - R$ of the refraction values (in arcsec), calculated for the ellipsoidal model, as compared to corresponding values for the spherical model.

*Table 2*
The deviation $\Delta(z,\varphi)$ of refraction value for the ellipsoidal model as compared to the value for spherical model

| $z$ \ $\varphi$ | 0 | 10 | 20 | 30 | 40 | 45 | 50 | 60 |
|---|---|---|---|---|---|---|---|---|
| 70 | 0.000 | 0.067 | 0.126 | 0.169 | 0.192 | 0.195 | 0.192 | 0.169 |
| 71 | 0.000 | 0.082 | 0.154 | 0.207 | 0.235 | 0.239 | 0.236 | 0.208 |
| 72 | 0.000 | 0.101 | 0.190 | 0.256 | 0.291 | 0.296 | 0.292 | 0.257 |
| 73 | 0.000 | 0.126 | 0.237 | 0.320 | 0.364 | 0.370 | 0.365 | 0.321 |
| 74 | -0.000 | 0.159 | 0.300 | 0.405 | 0.462 | 0.469 | 0.462 | 0.407 |
| 75 | -0.000 | 0.204 | 0.385 | 0.520 | 0.593 | 0.602 | 0.593 | 0.523 |
| 76 | -0.000 | 0.265 | 0.501 | 0.678 | 0.772 | 0.785 | 0.773 | 0.681 |
| 77 | -0.010 | 0.349 | 0.663 | 0.898 | 1.024 | 1.041 | 1.026 | 0.904 |
| 78 | -0.010 | 0.468 | 0.894 | 1.213 | 1.385 | 1.408 | 1.387 | 1.221 |
| 79 | -0.020 | 0.641 | 1.231 | 1.674 | 1.913 | 1.945 | 1.917 | 1.687 |
| 80 | -0.050 | 0.892 | 1.732 | 2.365 | 2.707 | 2.753 | 2.714 | 2.386 |
| 81 | -0.110 | 1.261 | 2.492 | 3.423 | 3.929 | 3.998 | 3.941 | 3.458 |

We also present the graphic of the deviation $\Delta(z,\varphi)$ for $\varphi$ = 45°, latitude for which the deviation is more pronounced.

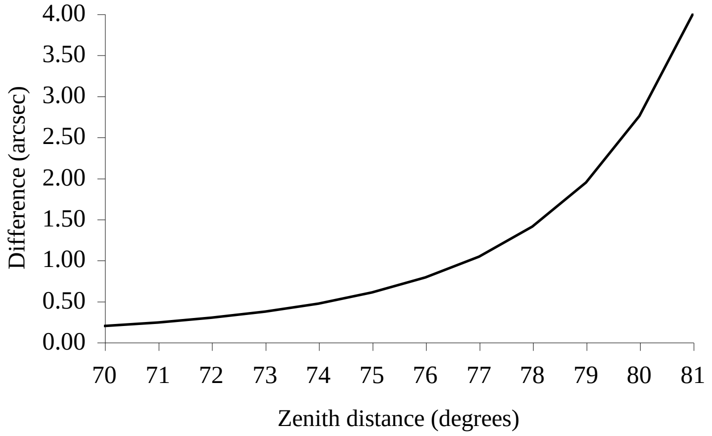


Fig.2 – Difference of the refraction values for the latitude of 45°.

It is ascertained that the deviations have a systematical character. Therefore, the results obtained using the ellipsoidal model of the atmosphere prove that, at least for the meridian plane of the observational place, the refraction is greater than the one calculated for the spherical model. The results are in accordance with those obtained for zenithal distances below 70°.

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