THE CALCULATION OF THE ASTRONOMICAL REFRACTION FOR THE ELLIPSOIDAL ATMOSPHERE OF THE EARTH

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Abstract. The astronomical refraction for the ellipsoidal atmosphere of the Earth is calculated, when the light ray is situated in the meridian plane of the place. It is shown that the Laplace-Oriani theorem is also valid for the considered model. By comparing these results to the ones of the spherical model, it is ascertained that the refraction calculated for the ellipsoidal model is greater than the one for the spherical model, the difference increasing with the zenith distance and depending on the latitude of the place.

Key words: astrometry – astronomical refraction – atmosphere models.

1. INTRODUCTION

Even from the beginning of the XX^{th} century, a model of the atmosphere of the Earth has been searched, which may allow the calculation of the astronomical refraction with a better approximation than in the case of the classical model of the atmosphere, composed of concentric spherical layers. An initial model was proposed by Harzer (1922,1924), which took into consideration the Earth as being a revolution ellipsoid, and the atmosphere as being composed of optic surfaces. This model was improved by Sergienko (1979), Shabelnikov (1983), Yatsenko and Teleki (Yatsenko and Teleki 1985;Yatsenko 1995), the calculation still being time-taking. A different model, geometrically simple, was provided by the first author of this paper (Mihăilă 1973), who considered the atmosphere as an ellipsoidal layer.

We add also that a first attempt to consider the optical surfaces as spheroidal was made by Radau (1882, 1889). He shows that the effect of the inclination of the surfaces with respect to the spherical surfaces is small and increases in the neighbourhood of the horizon. On the other hand, afterwards Newcomb (1906) considered the equipotential surfaces of the atmosphere as ellipsoidal. But he

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replaces the radius of curvature of the surface at each point of the trajectory by the sum of the radius of curvature of the Earth ellipsoid at the observational place and the height of the considered point. For this reason the effect of curvature is diminished, and Newcomb concludes that this can be neglected.

In this paper, making use of the refraction integral deduced by the first author, we shall present the calculation of the refraction for the ellipsoidal model, when the refracted ray is situated in the meridian plane of the observational place, and the obtained results will be compared to those from the spherical model, for zenith distances up to 70°.

We specify that the observations made on the Earth for the precise determination of angular coordinates of celestial bodies are carried out through observation of their passage over the meridian of the place. Only in this case the path of the light ray is a plane curve.

2. REFRACTION SERIES

It is assumed that the atmosphere of the Earth is composed of ellipsoidal layers of constant density. The separation surfaces of the layers are homothetic ellipsoids, one of these being the ellipsoid of the Earth. The intersections of these surfaces with the meridian plane of the observational place will be homothetic concentric ellipses. For this model, the first author obtained the following formula for the integral of refraction (Mihăilă 1973)

$$R = \int_1^{n_0} \frac{r_0 n_0 \sin z}{\sqrt{r^2 n^2 - r_0^2 n_0^2 \sin^2 z}} \frac{\mathrm{d}n}{n}, \tag{1}$$

where z is the observed zenith distance, r is the radius of curvature of the ellipse which passes through the current point P on the light ray trajectory in the atmosphere, and n the refraction index of the atmospherical layer in which the considered point is situated. The suffix zero signifies the data from the observational place P_0. The integral has the same form as in the case of the spherical model, with the simple difference that it contains the radius of curvature instead of the geocentric distance. The relation (1) can also be written as

$$R = \int_1^{n_0} \frac{\frac{r_0}{r}\frac{n_0}{n}\tan z}{\sqrt{1 + \left(1 - \frac{r_0^2}{r^2}\frac{n_0^2}{n^2}\right)\tan^2 z}} \frac{\mathrm{d}n}{n}. \tag{2}$$

We introduce the following notation

$$2u_\varphi = 1 - \frac{n_0^2}{n^2}\frac{r_0^2}{r^2}. \tag{3}$$

For $z \le 70°$, $2u_\varphi$ takes smaller values than $2.5 \cdot 10^{-2}$, hence the following series expansion can be used

$$\left(1 + 2u_\varphi \tan^2 z\right)^{-\frac{1}{2}} = 1 - u_\varphi \tan^2 z + \frac{3}{2}u_\varphi^2 \tan^4 z - \frac{5}{2}u_\varphi^3 \tan^6 z + \ldots. \tag{4}$$

Performing the replacement in relation (2), we obtain

$$R_\varphi = \tan z \int_1^{n_0} \frac{r_0}{r} \mathrm{d}\left(\frac{n_0}{n}\right) - \tan^3 z \int_1^{n_0} \frac{r_0}{r} u_\varphi \mathrm{d}\left(\frac{n_0}{n}\right) + \frac{3}{2}\tan^5 z \int_1^{n_0} \frac{r_0}{r} u_\varphi^2 \mathrm{d}\left(\frac{n_0}{n}\right) - \ldots.$$

From this series, only the first two terms have significance for z ≤ 70°, with an error less than 0″.001. Therefore, we can write

$$R_\varphi = A_{0\varphi} \tan z - A_{1\varphi} \tan^3 z, \tag{5}$$

where

$$A_{0\varphi} = \int_1^{n_0} \frac{r_0}{r} \mathrm{d}\left(\frac{n_0}{n}\right), \quad A_{1\varphi} = \int_1^{n_0} \frac{r_0}{r} u_\varphi \mathrm{d}\left(\frac{n_0}{n}\right). \tag{6}$$

The ratio $\frac{r_0}{r}$ is a complicated function of a - the semimajor axis of the ellipse passing through the current point P of the ray trajectory and of φ - the geodetic latitude of point P. Proceeding as in the case of the spherical model (see, e. g., Dinulescu, 1967), we shall use the following transformation

$$\frac{r_0}{r} = c_\varphi (1 - s),\; s = 1 - \frac{a_0}{a}. \tag{7}$$

Using the expression of the radius of curvature of the ellipse, this can be written

$$\frac{r_0}{r} = \frac{a_0}{a}\left(\frac{1 - e^2 \sin^2 \varphi}{1 - e^2 \sin^2 \varphi_0}\right)^{\frac{3}{2}} = \frac{a_0}{a}\left[\left(1 - e^2 \sin^2 \varphi\right)\left(1 - e^2 \sin^2 \varphi_0\right)^{-1}\right]^{\frac{3}{2}}. \tag{8}$$

The product $e^2 \sin^2 \varphi_0$ is very small, between 0 and $6.4 \cdot 10^{-3}$. Therefore, developing in series and keeping the first two terms, we obtain

$$\frac{r_0}{r} = \frac{a_0}{a}\left[1 + e^2\left(\sin^2 \varphi_0 - \sin^2 \varphi\right)\right]^{\frac{3}{2}}. \tag{9}$$

Finally, we obtain

$$\frac{r_0}{r} = \frac{a_0}{a}\left[1 + e^2 \sin\left(\varphi_0 + \varphi\right)\sin\left(\varphi_0 - \varphi\right)\right]^{\frac{3}{2}}. \tag{10}$$

Because the product $\left|e^2 \sin(\varphi_0 - \varphi)\sin(\varphi_0 + \varphi)\right| \le 6.4 \cdot 10^{-3}$, developing in series and retaining the first two terms, we can write

$$\frac{r_0}{r} = \frac{a_0}{a}\left[1 + \frac{3}{4}e^2\left(\cos 2\varphi - \cos 2\varphi_0\right)\right]. \tag{11}$$

Because the geodetic latitude variation of the current point is small, the value of $\cos 2\varphi$ can be averaged by the formula

$$\overline{\cos 2\varphi} = \frac{1}{2}\left(\cos 2\varphi_i + \cos 2\varphi_0\right), \tag{12}$$

where φ_i is the geodetic latitude of point P_i, where the radiation penetrates the terrestrial atmosphere at the upper limit. This latitude is obtained approximately, determining the position of the point situated at the intersection of the tangent to the trajectory in P_0 with the ellipse corresponding to the upper limit of the atmosphere . Because only a thin layer of the height $h \approx 80\,\text{km}$ produces a sensible refraction effect (Dinulescu, 1967), we assume for more accuracy that the upper limit of the atmosphere is situated at $h \approx 100\,\text{km}$.

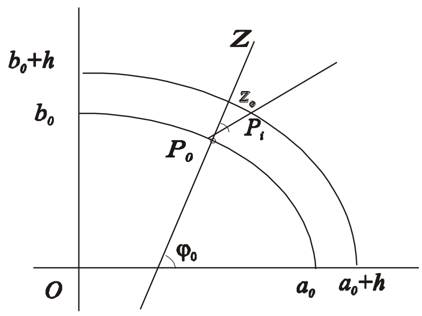

Fig. 1 – The configuration from the meridian plane of the place

The coordinates of the point of entrance in the atmosphere are given by the system

$$\begin{cases} y - y_0 = (x - x_0)\tan(\varphi_0 - z), x_0 = \dfrac{a_0 \cos\varphi_0}{\sqrt{1 - e^2 \sin^2 \varphi_0}}, y_0 = \dfrac{a_0\left(1 - e^2\right)\sin\varphi_0}{\sqrt{1 - e^2 \sin^2 \varphi_0}}, \\ \left(\dfrac{x}{a_0 + h}\right)^2 + \left(\dfrac{y}{b_0 + h}\right)^2 = 1, \end{cases} \tag{13}$$

where a_0 and b_0 are the semiaxes of the terrestrial meridian. If (x_i, y_i) is the solution of the system (13), then $\varphi_i = \arctan \dfrac{y_i}{x_i}\left(\dfrac{a_0 + h}{b_0 + h}\right)^2$. Then the expression of the coefficient c_φ from (7) becomes

$$c_{\varphi} = 1 + \frac{3}{4} e^{2} \left(\overline{\cos 2\varphi} - \cos 2\varphi_{0} \right). \tag{14}$$

Replacing the ratio $\frac{r_0}{r}$ from the relation (7) in the integrand of $A_{0\varphi}$, we obtain

$$A_{0\varphi} = c_{\varphi} \int_{1}^{n_0} (1 - s) \mathrm{d}\left(\frac{n_0}{n} \right) = c_{\varphi} \left[\int_{1}^{n_0} \mathrm{d}\left(\frac{n_0}{n} \right) - \int_{1}^{n_0} s \mathrm{d}\left(\frac{n_0}{n} \right) \right] = c_{\varphi} \left[(n_0 - 1) - I \right],$$

where

$$I = \int_{1}^{n_0} s \mathrm{d}\left(\frac{n_0}{n} \right). \tag{15}$$

This integral can be written

$$I = \int_{1}^{n_0} s \, \mathrm{d}\left(\frac{n_0}{n} \right) = \int_{1-n_0}^{0} s \, \mathrm{d}\left(\frac{n_0}{n} - n_0 \right) = - \int_{0}^{\frac{h}{a_0 + h}} \left(\frac{n_0}{n} - n_0 \right) \mathrm{d}s.$$

Using the variable $\frac{a_0}{a}$, we have

$$I = \int_{0}^{\frac{h}{a_0 + h}} \frac{n_0}{n} (1 - n) \mathrm{d}\left(\frac{a_0}{a} \right). \tag{16}$$

For the dependence between the refraction and density indices, we shall use, for simplicity, the Gladstone relation

$$\frac{n_0 - 1}{n - 1} = \frac{\delta_0}{\delta}, \tag{17}$$

where δ is the density.

On the other hand, the dependence between the density of the atmosphere and the pressure of the atmosphere is given by the equation of hydrostatic equilibrium

$$\mathrm{d}p = -g(a) \delta \mathrm{d}a, \tag{18}$$

where $g(a)$ is the acceleration of gravity at $a - a_0$ altitude. This value can be expressed, with a good approximation, by means of the equatorial acceleration of gravity at the surface of the Earth. Indeed,

$$\frac{g(a)}{g_{ec}} = \left(\frac{a_0}{a} \right)^2 (1 - q + q_0) = \left(\frac{a_0}{a} \right)^2 \left[1 + \frac{\omega^2}{fM} \left(a_0^3 - a^3 \right) \right] = \gamma_a \left(\frac{a_0}{a} \right)^2, \tag{19}$$

where $q=\frac{\omega^2 a^3}{fM}$, $\gamma_a=1+\frac{\omega^2}{fM}\left(a_0^3-a^3\right)$, ω being the angular velocity of rotation of the Earth, and fM the geocentric gravitational constant. When a varies, γ_a takes values between 0.99983 and 1, therefore (19) can be written

$$\frac{g(a)}{g_{ec}}=\left(\frac{a_0}{a}\right)^2. \tag{20}$$

Replacing the expression for $g(a)$ in the equation (18), we obtain

$$\mathrm{d}p=g_{ec}a_0\delta\mathrm{d}\left(\frac{a_0}{a}\right). \tag{21}$$

Therefore

$$\mathrm{d}\left(\frac{a_0}{a}\right)=\frac{\mathrm{d}p}{g_{ec}a_0\delta}=\frac{p_0}{g_{ec}a_0\delta_0}\frac{\delta_0}{\delta}\frac{\mathrm{d}p}{p_0}. \tag{22}$$

Let β_{ec} be the first ratio of the last member of the equality and let us consider β_{ec} = 0.0013260. Using (18), we have

$$\mathrm{d}\left(\frac{a_0}{a}\right)=\beta_{ec}\frac{n_0-1}{n-1}\frac{\mathrm{d}p}{p_0}. \tag{23}$$

Returning to the integral (16), this can be written

$$I=\beta_{ec}\left(n_0-1\right)\int_0^{p_0}\frac{n_0}{n}\frac{\mathrm{d}p}{p_0}. \tag{24}$$

Approximating the ratio $\frac{n_0}{n}$ by 1 and denoting n_0-1 by α_0, we obtain

$$I=\beta_{ec}\alpha_0. \tag{25}$$

Therefore, the first coefficient has the expression

$$A_{0\varphi}=c_\varphi\alpha_0\left(1-\beta_{ec}\right). \tag{26}$$

To determine the expression of the coefficient $A_{1\varphi}$, taking into account the relations (3) and (7), the integrand becomes, succesively

$$c_\varphi(1-s)u_\varphi=\frac{c_\varphi}{2}(1-s)\left[1-c_\varphi^2(1-s)^2\left(\frac{n_0}{n}\right)^2\right]=$$

.

$$=\frac{c_\varphi}{2}\left[1-s-c_\varphi^2(1-s)^3\left(\frac{n_0}{n}\right)^2\right]=$$

$$=\frac{c_\varphi}{2}\left[1-s-c_\varphi^2\left(\frac{n_0}{n}\right)^2+3c_\varphi^2 s\left(\frac{n_0}{n}\right)^2-3c_\varphi^2 s^2\left(\frac{n_0}{n}\right)^2+s^3c_\varphi^2\left(\frac{n_0}{n}\right)^2\right].$$

Neglecting s^2 and considering $s\left(\frac{n_0}{n}\right)^2 \approx s,$ the expression becomes

$$c_\varphi(1-s)u_\varphi=\frac{c_\varphi}{2}\left[1-s-c_\varphi^2\left(\frac{n_0}{n}\right)^2+3sc_\varphi^2\right]. \tag{27}$$

Integrating from 1 to n_0 and using the integral (25), we obtain

$$A_{1\varphi}=\frac{c_\varphi}{2}\int_1^{n_0}\left[1+\left(3c_\varphi^2-1\right)s-c_\varphi^2\left(\frac{n_0}{n}-1\right)^2-2c_\varphi^2\left(\frac{n_0}{n}-1\right)-c_\varphi^2\right]\mathrm{d}\left(\frac{n_0}{n}\right),$$

or

$$A_{1\varphi}=\frac{c_\varphi}{2}\left[-\alpha_0\left(c_\varphi^2-1\right)+\alpha_0\beta_{ec}\left(3c_\varphi^2-1\right)-\frac{c_\varphi^3}{3}\alpha_0^3-c_\varphi^2\alpha_0^2\right]. \tag{28}$$

In the obtained expression α_0^3 can be neglected, because the error introduced in such a way is less than 0″.005. Consequently, we can write

$$A_{1\varphi}=\frac{c_\varphi\alpha_0}{2}\left[1+\left(3c_\varphi^2-1\right)\beta_{ec}-c_\varphi^2 n_0\right]. \tag{29}$$

Substituting the expressions (26) and (29) for $A_{0\varphi}$ and $A_{1\varphi}$ in (5), we come to the conclusion that for $z \le 70^\circ$ the Laplace – Oriani theorem is also valid for the ellipsoidal model. Therefore for zenith distances below 70° the astronomical refraction is independent of the structure of the atmosphere. Evidently, for the excentricity $e=0$, c_φ becomes 1, and the obtained formula is reduced to the Laplace formula.

3. RESULTS

On the hypothesis in which the atmosphere of the Earth is composed of concentric spherical layers, the astronomical refraction can be calculated, for zenith distances $z \le 70°$, using the Laplace formula

$$R=\alpha_0(1-\beta_0)\tan z-\alpha_0\left(\beta_0-\frac{\alpha_0}{2}\right)\tan^3 z. \tag{30}$$

For this calculation, we shall consider the values $\alpha_0=0.0002927$ and $\beta_0=0.0013238$.

In the present study the atmosphere of the Earth was considered to be composed of ellipsoidal layers, of the same eccentricity. From the geometrical point of view, this model represents a better approximation of the Earth and of the terrestrial atmosphere. Consequently, the obtained refraction formula is more complicated. Naturally, there arises the problem of comparing the values obtained for the refraction to those given by the spherical model.

The refraction formula in the case of the ellipsoidal model is

$$R_\varphi = c_\varphi \alpha_0 (1 - \beta_{ec}) \tan z - \frac{c_\varphi}{2} \alpha_0 \left[1 + \left(3c_\varphi^2 - 1\right)\beta_{ec} - c_\varphi^2 n_0\right] \tan^3 z, \tag{31}$$

where c_φ is given by the formula (14). For calculation we shall adopt the values $\alpha_0 = 0.0002927$, $\beta_{ec} = 0.0013260$.

In Table 1 are presented the values of the refraction *R* (in arcsec), for a few zenith distances (in degrees), calculated using the Laplace formula.

Table 2 presents the differences $\Delta(z,\varphi) = R_\varphi - R$ of the refraction values (in arcsec), calculated for the ellipsoidal model and for the spherical model, as a function of the zenith distance and latitude (in degrees).

Figures 2 – 4 represent the differences $\Delta(z,\varphi)$ for the geodetic latitudes of 30°, 45° and 60°.

It is ascertained that the differences have a systematic character. Beginning with 50° zenith distance for any latitude (excepting zero latitude) the differences become approximately ≥ 0″.01. The maximum values of the differences are ascertained for the latitude of 45°. Therefore, the results obtained using the atmospherical model proposed in this paper prove that, at least for the meridian plane of the obsevational place, the refraction is greater than the one calculated in the case of the spherical atmosphere of the Earth.

Table 1

Refraction for the spherical model

z	R	z	R
0	0	60	104.063
20	21.942	62	112.923
30	34.797	64	123.008
40	50.551	66	134.617
45	60.223	68	148.154
50	71.735	70	164.181
55	85.901		

Table 2

The differences $\Delta(z,\varphi)$ of the refraction value for the ellipsoidal model as compared to the value for the spherical model

z \ φ	0	10	20	30	40	45	50	60
0	0	0	0	0	0	0	0	0
20	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001
30	0.000	0.001	0.001	0.002	0.002	0.002	0.002	0.002
40	0.000	0.002	0.003	0.005	0.005	0.005	0.005	0.005
45	0.000	0.003	0.006	0.008	0.009	0.009	0.009	0.008
50	-0.001	0.005	0.010	0.013	0.015	0.015	0.015	0.013
55	-0.001	0.009	0.017	0.023	0.027	0.027	0.027	0.024
60	-0.002	0.016	0.033	0.045	0.052	0.053	0.052	0.046
62	-0.003	0.021	0.043	0.060	0.069	0.070	0.069	0.062
64	-0.004	0.029	0.059	0.082	0.094	0.096	0.095	0.084
66	-0.006	0.040	0.082	0.113	0.130	0.133	0.131	0.117
68	-0.010	0.057	0.116	0.160	0.185	0.189	0.187	0.166
70	-0.014	0.083	0.169	0.234	0.271	0.277	0.274	0.244

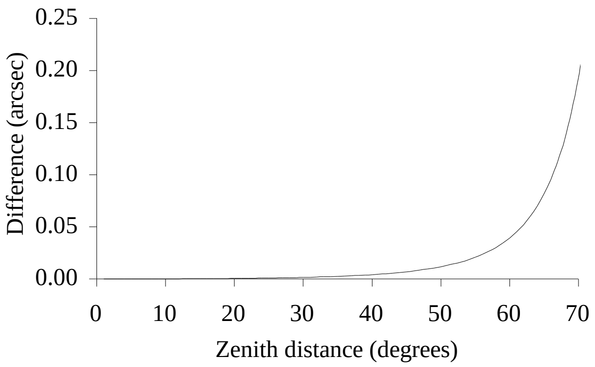

Fig. 2 – Difference of the refraction values for the latitude of 30°.

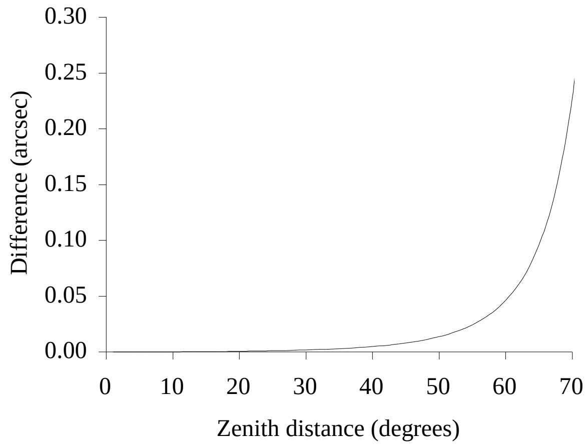

Fig. 3 – Difference of the refraction values for the latitude of 45°.

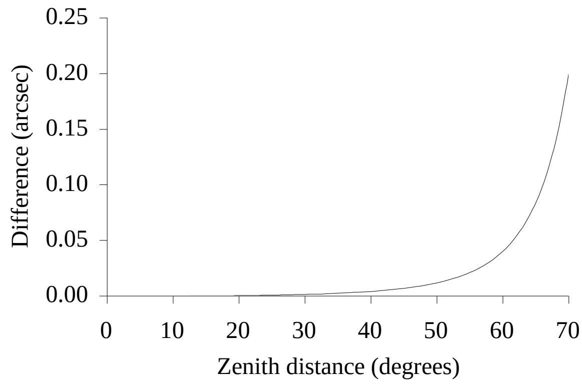

Fig. 4 – Difference of the refraction values for the latitude of 60°.

The found differences are due to the c_φ factor, which in the case of the spherical model becomes 1. This factor depends on the latitude and the zenith distance (Table 3).

Table 3
The values of c_φ factor (z and φ in degrees)

z \ φ	0	30	45	60	90
0	1	1	1	1	1
20	1	1.000024	1.000028	1.000024	1
30	1	1.000039	1.000045	1.000039	1
40	0.999999	1.000056	1.000065	1.000056	1.000001
45	0.999999	1.000066	1.000077	1.000067	1.000001
50	0.999998	1.000079	1.000092	1.00008	1.000002
55	0.999998	1.000094	1.000109	1.000095	1.000002
60	0.999997	1.000112	1.000131	1.000115	1.000003
62	0.999996	1.000121	1.000142	1.000125	1.000004
64	0.999995	1.000132	1.000154	1.000136	1.000005
66	0.999994	1.000143	1.000168	1.000148	1.000006
68	0.999993	1.000156	1.000184	1.000162	1.000007
70	0.999992	1.000171	1.000202	1.000179	1.000008

On the other hand, for a given latitude, c_φ is an increasing function of altitude and zenith distance. For the latitude $\varphi = 45^\circ$, the function $f(h) = (c_\varphi - 1)10^4$ is represented in Fig. 5.

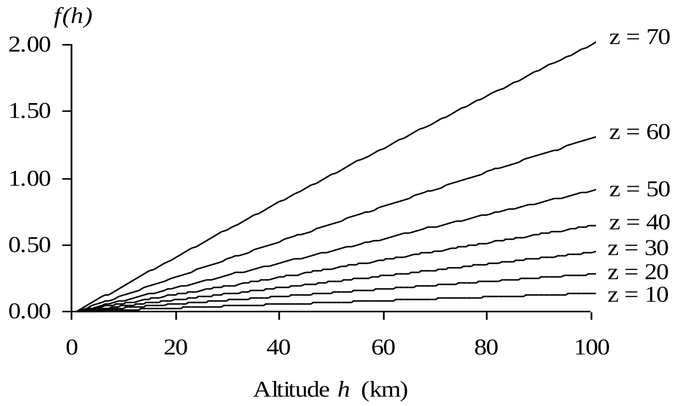

Fig. 5 – Dependence of the c_φ factor of altitude for $\varphi = 45^\circ$

Obviously, the values of c_φ make physical sense only to height of the layer which produces a sensible refraction effect. For larger heights, although c_φ increases, the effect is insignificant, because refraction tends to zero.

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REFERENCES

Dinulescu, N. I.: 1967, *Astronomical Refraction*, Editura Academiei, Bucharest. (Romanian)
Harzer, P.: 1922, *Publikation der Sternwarte in Kiel*, **13**, 2.
Harzer, P.: 1924, *Publikation der Sternwarte in Kiel*, **14**, 1.
Mihăilă, I.: 1973, *St. Cerc. Astron*., **18**, 187.
Newcomb, S.: 1906, *A Compendium on Spherical Astronomy*, Macmillan, New York.
Radau, R.: 1882, *Ann. Obs. Paris*, **XVI**, B1.
Radau, R.: 1889, *Ann. Obs. Paris*, **XIX**, G1.
Sergienko, V.I.: 1979, *Astron. Zh*., **56**, 672.
Shabelnikov, A.V.: 1983, *Pis'ma Astron. Zh*., **9**, 695.
Yatsenko, A. Yu.: 1995, *Astron. Astrophys. Suppl*., **111**, 579.
Yatsenko A. Yu., Teleki, G.: 1985, *Bull. Observ. Astron. Beograd*, **135**, 1.

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