RECURRENT POWER SERIES SOLUTION OF THE *n*-BODY PROBLEM ASSOCIATED TO A QUASIHOMOGENEOUSE POTENTIAL

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Abstract. Using Steffensen's method, a recurrent power series solution is given for the n-body problem associated to a quasihomogeneous potential of form $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leq a \leq b$. The application to numerical integration is also pointed out.

Key words: celestial mechanics, generalized fields, power series, numerical integration.

1. INTRODUCTION

In 1956-1957 a few articles have been published by Steffensen (1956a, 1956b, 1957), describing the solution of both the restricted and the general three-body problem in term of power series in time. The method proposed by Steffensen is particularly well adopted to computers. The method is made practical by the introduction of a certain number of auxiliary dependent variables, wich transform the system of differential equations where all denominators have been remuved, as vell as the powers r^3. Steffensen calls his systemof 'second degree', because in the final form, only products of two dependent variables appear. This form is particularly well-prepared for the substitution of power series and the identification of equal powers in t. In several of his papers Steffensen has also given convergence criteria for the series. The application of the series is particularly interesting because the square roots are completly avoided in the computations and the number of divisions is reduced to a minimum. The reason why the method is so well adopted to automatic computers is that the calculation of all the coefficients of the power series is done in a recurrent way; for each order, the coefficients are functions of all the precedingly computed coefficients.

This method has been effectively used by several authors, such as Rabe (1961), Deprit (1965), Broucke (1971), Pál and Szenkovits (1996) for the numerical integration of the restricted three-body problem and the general n-body problem on computers, and it appears that the results are superior both in speed and in precision, to those obtained with most of the classical numerical integration methods.

The goal of this paper is to give the recurrent power series solution for the n-body problem associated to a quasihomogeneous potential of form $W = U + V$, where U and V are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \leq a \leq b$.

2. THE *n*-BODY PROBLEM ASSOCIATED TO A QUASIHOMOGENEOUS POTENTIAL

The n-body problem associated to the quasihomogeneous potential can be formulated as follows (see Diacu, 1996). Consider n particles of masses $m_i > 0$ in the Euclidean space E^3, having coordinates $\mathbf{q_i} = (q_i^1, q_i^2, q_i^3)$, $i = 1, 2, ..., n$, in an inertial reference system. Let $\mathbf{q} = (\mathbf{q_1}, \mathbf{q_2}, ..., \mathbf{q}_n) \in \mathbf{R}^{3n}$ be the configuration of the system of particles and define the *quasihomogeneous* potential $W = U + V$, where

$$U : \mathbf{R}^{3n} \setminus \Delta \to \mathbf{R}_+,\ U(\mathbf{q}) = \sum_{1\le i<j\le n} \alpha\left(m_i, m_j\right) q_{ij}^{-a},$$
$$V : \mathbf{R}^{3n} \setminus \Delta \to \mathbf{R}_+,\ V(\mathbf{q}) = \sum_{1\le i<j\le n} \beta\left(m_i, m_j\right) q_{ij}^{-b}$$

are homogeneous functions of degree $-a$ and $-b$ respectively, with $1 \le a \le b$. In these potentials $q_{ij} = |\mathbf{q_i} - \mathbf{q_j}|$ is the Euclidean distance between particles i and j, Δ denotes the collision-ejection set

$$\Delta = \bigcup_{1\le i<j\le n} \left\{\mathbf{q} \middle| \mathbf{q_i} = \mathbf{q_j}\right\},$$

and α, β are symmetric positive functions of the masses, i.e. such that $\alpha(m_i, m_j) = \alpha(m_j, m_i) = \alpha_{ij} > 0$, and $\beta(m_i, m_j) = \beta(m_j, m_i) = \beta_{ij} > 0$, for all $1 \le i < j \le$ n.

The equations of motion are given by the system

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}), \end{cases} \tag{1}$$

where $M = \mathrm{diag}(m_1, m_1, m_1, m_2, m_2, m_2, ..., m_n, m_n, m_n)$, $\nabla = \left(\partial/\partial\mathbf{q}_1, \partial/\partial\mathbf{q}_2, \ldots, \partial/\partial\mathbf{q}_n\right)$ is the gradient operator and $\mathbf{p} = M\dot{\mathbf{q}}$, $\mathbf{p} = (\mathbf{p_1}, \mathbf{p_2}, ..., \mathbf{p}_n) \in \mathbf{R}^{3n}$ denotes the *momentum* of the system. In case $a = b = 1$ and $\alpha(m_i, m_j) = \beta(m_i, m_j) = (G/2)\, m_i m_j$, where G is the gravitational constant, we are in the classical Newtonian n-body problem.

3. RECURRENT POWER SERIES SOLUTION

The equations of the motions (1) are equivalents with the next ones:

$$\begin{cases} m_i \dot{\mathbf{q}}_i = \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \displaystyle\sum_{\substack{j=1 \\ j\ne i}}^{n} \left(a\alpha_{ij} q_{ij}^{-a-2} + b\beta_{ij} q_{ij}^{-b-2}\right)\left(\mathbf{q}_j - \mathbf{q}_i\right), \end{cases} \quad i = 1, 2, ..., n \tag{2}$$

The central idea of Steffensen's method is to introduce auxiliary variables, wich help us to eliminate quantities $q_{ij}^{a+2}, q_{ij}^{b+2}$ from the denominators of the right hand side expressions in equations (2). Let it be:

$$\begin{cases} a_{ij} = q_{ij}^{-a-2} \\ b_{ij} = q_{ij}^{-b-2} \end{cases}, \quad 1 \le i < j \le n \tag{3}$$

Using these new variables introduced in (3), the equatios of mevement (2) have the new form

$$\begin{cases} m_i \dot{\mathbf{q}}_i = \mathbf{p}_i \\ \dot{\mathbf{p}}_i = \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left(a\alpha_{ij} a_{ij} + b\beta_{ij} b_{ij}\right)\left(\mathbf{q}_j - \mathbf{q}_i\right), \end{cases} \quad i = 1, 2, ..., n \tag{4}$$

Equations (4) can be completed with the next relations betwen old and new variables:

$$\begin{cases} q_{ij}\dot{q}_{ij} = \left(\mathbf{q}_j - \mathbf{q}_i\right)\left(\dot{\mathbf{q}}_j - \dot{\mathbf{q}}_i\right) \\ q_{ij}\dot{a}_{ij} = -(a+2)a_{ij}\dot{q}_{ij} \\ q_{ij}\dot{b}_{ij} = -(b+2)b_{ij}\dot{q}_{ij} \end{cases}, \quad 1 \le i < j \le n \tag{5}$$

Equations (4) and (5) constitute a differential system wich determine variables $\mathbf{q_i}$, $\mathbf{p_i}$, q_{ij}, a_{ij}, b_{ij}, $1 \le i < j \le n$. This system can not be integrated with exact methods. We can obtain the solutions of this system using power series:

$$\mathbf{q}_i = \sum_{k=1}^{\infty} \mathbf{Q}_{ik} t^{k-1}, \quad \mathbf{p}_i = \sum_{k=1}^{\infty} \mathbf{P}_{ik} t^{k-1},$$

$$q_{ij} = \sum_{k=1}^{\infty} Q_{ijk} t^{k-1}, \quad a_{ij} = \sum_{k=1}^{\infty} A_{ijk} t^{k-1}, \quad b_{ij} = \sum_{k=1}^{\infty} B_{ijk} t^{k-1}, \tag{6}$$

where coefficients $\mathbf{Q_{ik}}$, $\mathbf{P_{ik}} \in \mathbf{R}^3$, Q_{ijk}, A_{ijk}, $B_{ijk} \in \mathbf{R}$, $1 \le i < j \le n$, $k \ge 1$ will be determined.

Substituting the power series (6) in equations (4, 5) after identification of equal powers in t, one obtains the next recurrence relations, to determine the unknown coefficients:

$$\begin{aligned}
& k m_i \mathbf{Q}_{i,k+1} = \mathbf{P}_{ik} \\
& k\mathbf{P}_{i,k+1} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left[a\alpha_{ij} \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} A_{ijp}\left(\mathbf{Q}_{jq} - \mathbf{Q}_{iq}\right) + b\beta_{ij} \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} B_{ijp}\left(\mathbf{Q}_{jq} - \mathbf{Q}_{iq}\right) \right] \\
& kQ_{ij1} Q_{ij,k+1} = -\sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} qQ_{ijp} Q_{ij,q+1} + \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} q\left(\mathbf{Q}_{jp} - \mathbf{Q}_{ip}\right)\left(\mathbf{Q}_{j,q+1} - \mathbf{Q}_{i,q+1}\right) \\
& kQ_{ij1} A_{ij,k+1} = -\sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} qQ_{ijp} A_{ij,q+1} - \left(a+2\right) \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} qA_{ijp} Q_{ij,q+1} \\
& kQ_{ij1} B_{ij,k+1} = -\sum_{\substack{p+q=k+1 \\ p=2,\dots,k}} qQ_{ijp} B_{ij,q+1} - \left(b+2\right) \sum_{\substack{p+q=k+1 \\ p=1,\dots,k}} qB_{ijp} Q_{ij,q+1}
\end{aligned} \tag{7}$$

Start coefficients $\mathbf{Q_{i1}}$, $\mathbf{P_{i1}} \in \mathbf{R}^3$, Q_{ij1}, A_{ij1}, $B_{ij1} \in \mathbf{R}$, $1 \le i < j \le n$ are obtained from the initial conditions given for $t = 0$, $\mathbf{q_{i0}} = \mathbf{q_i}(0)$, $\mathbf{p_{i0}} = \mathbf{p_i}(0) \in \mathbf{R}^3$, $i = 1, 2, ..., n$. These coefficients are:

$$\begin{gathered} \mathbf{Q_{i1}} = \mathbf{q_{i0}}, \mathbf{P_{i1}} = \mathbf{p_{i0}}, \ i = 1, 2, ..., n \\ Q_{ij1} = |\mathbf{Q_{i1}} - \mathbf{Q_{j1}}|, \ A_{ij1} = Q_{ij1}^{-a-2}, \ B_{ij1} = Q_{ij1}^{-b-2}, \ 1 \le i < j \le n. \end{gathered} \tag{8}$$

Start coefficients (8) calculated, one computes all coefficients of Taylor series (6), with the index $k + 1$, step by step using the recurent relations (7).

5. APLICATIONS TO NUMERICAL INTEGRATION

This method is well adopted to automatic computers, since calculation of all the coefficients of the power series is done in a recurrent way; for each order, the coefficients are functions of all the precedingly computed coefficients. In approximate numerical solutions the coefficients are calculated until an order N, suitable chosen. To verify computetional accuraty, one can uses the *energy integral*

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h$$

where $T : \mathbf{R}^{3n} \to [0. +\infty)$, $T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{n} m_i^{-1}|\mathbf{p_i}|^2$ is the *kinetic energy* and h is the *energy constant*; and the *momentum integral*

$$\sum_{i=1}^{n}[\mathbf{q_i}, \mathbf{p_i}] = \mathbf{c} \text{ (constant).}$$

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