STOCHASTIC SET-UP OF THE TWO-BODY PROBLEM

CRISTINA STOICA

Institute for Gravitation and Space Sciences, 21, Ana Ipătescu Bvd., 71111 Bucharest, Romania

GH. STOICA

University of Bucharest, Mathematics Department, 14, Academiei St., 70109 Bucharest, Romania

Abstract. Using *Bismut's* approach to the stochastic canonical system, this paper applies stochastic methods to the concrete case of the perturbed two-body problem.

Key words: celestial mechanics — two-body problem

1. THEORETICAL CONSIDERATIONS

Let $(\vec{q}, \vec{p}) \in R^{2n}$, where $\vec{q} = (q_1, \ldots, q_n)$ is the generalized coordinate and $\vec{p} = (p_1, \ldots, p_n)$ is the generalized impulse of an n point-like masses system, whose motion is described by the canonical system:

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}, \tag{1}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}},$$

with obvious notations (see [1] *Arnold*, 1976).

In the sequel we shall suppose that the above system is randomly perturbed (e.g. by the high atmospherical perturbations), so let us consider a random potential of the form:

$$\mathcal{U}(t, \vec{q})\, \dot{B}_t(\omega), \tag{2}$$

where $B_t(\omega) : [0, \infty) \times \Omega \to \mathbf{R}$ is a standard *Brownian* motion process defined on a probability field $(\Omega, \mathcal{K}, P)$ endowed with the natural filtration of B and $\dot{B}_t(\omega)$ is the white noise measure, which controls the increments of B. The size of $\mathcal{U}(t, \vec{q})$ must be sufficiently small to dominate the "very large" size of the properly weighted increments of B (see [3] *Guikhman*, *Skorohod*, 1980).

From now on, if the quantities will be understood in the classical meaning, we shall specify it. If not, they will be understood in the above generalized (stochastic) meaning.

The new *Hamiltonian* takes the form:

$$\mathscr{H}(t, \vec{q}, \vec{p}) = H(t, \vec{q}, \vec{p}) + \mathscr{H}_1(t, \vec{q}, \vec{p}), \tag{3}$$

where its random part $\mathscr{H}_1$ is given by:

$$\mathscr{H}_1(t, \vec{q}, \vec{p}) = \mathscr{U}(t, \vec{q})\, \dot{B}_t(\omega). \tag{4}$$

From stochastic point of view $\mathscr{H}_1$ is defined as the *Legendre* transform of the *Lagrangean* obtained from the *Jacobi-Bellman* equation (see [2] *Bismut*, 1981) and takes the form:

$$\mathscr{H}_1(t, \vec{q}, \vec{p}) = \left\langle \vec{\sigma}(t, \vec{q}), \vec{p} + \frac{\partial W}{\partial \vec{q}}(t, \vec{q}) \right\rangle, \tag{5}$$

where $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathscr{C}^\infty(\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n)$ is bounded together with its derivatives on compact sets and $W = W(t, \vec{q})$ is a solution of the *Hamilton-Jacobi* equation associated to the classical canonical system (1):

$$\frac{\partial}{\partial t}[-W(t, \vec{q})] + H\left(t, \vec{q}, -\frac{\partial W}{\partial \vec{q}}(t, \vec{q})\right) = 0. \tag{6}$$

By fitting (4) and (5) one may determine $\vec{\sigma}$:

$$\sum_{i=1}^{n} \sigma_i \left(p_i + \frac{\partial W}{\partial q_i}\right) = \mathscr{U}(t, \vec{q})\, \dot{B}_t. \tag{7}$$

In classical meaning:

$$\vec{p} = -\frac{\partial W}{\partial \vec{q}}, \tag{8}$$

one can observe that in the left side brackets of (7) we have the difference between $p_i = p_i(t, \omega)$ as solution of the perturbed problem and $p_i = p_i(t)$ given by (8). So, if the impulse of the perturbed system reduces to the impulse of the unperturbed system, then the random potential must be zero.

2. STOCHASTIC CANONICAL SYSTEM

The canonical system for the new *Hamiltonian* $\mathscr{H}$ takes the following integral form:

$$\vec{q}(t, \omega) = \vec{q}(0, \omega) + \int_0^t \frac{\partial H}{\partial \vec{p}}(s, \vec{q}, \vec{p})\, \mathrm{d}s + \int_0^t \frac{\partial \mathscr{H}_1}{\partial \vec{p}}(s, \vec{q}, \vec{p})\, \mathrm{d}B_s, \tag{9}$$

$$\vec{p}(t, \omega) = \vec{p}(0, \omega) - \int_0^t \frac{\partial H}{\partial \vec{q}} q(s, \vec{q}, \vec{p})\, ds - \int_0^t \frac{\partial \mathscr{H}_1}{\partial \vec{q}} (s, \vec{q}, \vec{p})\, dB_s,$$

with the non-random initial data

$$\vec{q}(0, \omega) = \vec{q}_0, \vec{p}(0, \omega) = - \frac{\partial W}{\partial \vec{q}} (0, \vec{q}_0). \tag{10}$$

The integrals in (9) made with respect to the *Brownian* motion are in the sense of *Itô* (see [3] *Guikhman, Skorohod*, 1980).

The solution of (9)—(10) is given by (see [2] *Bismut*, 1981) :

$$\left(\vec{q}(t, \omega), - \frac{\partial W}{\partial \vec{q}} (t, \vec{q}(t, \omega))\right), \tag{11}$$

where $\vec{q}(t, \omega)$ is the unique solution of the following stochastic differential system :

$$\vec{q}(t, \omega) = \vec{q}_0 + \int_0^t \vec{f}(s, \vec{q}(s, \omega))\, ds + \int_0^t \vec{\sigma}(s, \vec{q}(s, \omega))\, dB_s(\omega), \tag{12}$$

with $\vec{f} \in \mathscr{C}^\infty(\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n)$ given by :

$$\vec{f}(t, \vec{q}) = \frac{\partial H}{\partial \vec{p}} \left(t, \vec{q}, - \frac{\partial W}{\partial \vec{q}} (t, \vec{q})\right) + \frac{1}{2} \frac{\partial \vec{\sigma}}{\partial \vec{q}} (t, \vec{q}) \cdot \vec{\sigma}(t, \vec{q}). \tag{13}$$

One can recognize that $\vec{f}$ and $\vec{\sigma}$ are the transfer, respectively the diffusion, coefficients of $\vec{q}$ in stochastic meaning. Under sufficiently large assumptions on $\vec{f}$ and $\vec{\sigma}$, the system (12) admits a continuous P-almost surely solution for which we have path uniqueness.

3. SOLUTION OF THE STOCHASTIC PERTURBED TWO-BODY PROBLEM

In the classical two-body problem, let be (r, φ) the polar coordinates and $\vec{q} = (q_1, q_2)$; $q_1 = r$, $q_2 = \varphi$, $\vec{p} = (p_1, p_2)$; $p_1 = \dot{r}$, $p_2 = r^2\dot{\varphi}$ the generalized coordinate, respectively the generalized impulse. Thus the classical *Hamiltonian* H equals :

$$H = \frac{1}{2}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) - \frac{\mu}{q_1},$$

where μ is the gravitational parameter.

By mechanical considerations it is natural to suppose that the random potential is of $O(\varepsilon)$ with respect to H, giving a small random perturbation. The random part of the *Hamiltonian* takes the form :

$$\mathscr{H}_1 = \varepsilon \mathscr{U}_1(t, \vec{q})\, \dot{B}_t(\omega).$$

Formula (7) yields to :

$$\mathscr{H}_1 = \varepsilon \left[\sigma_1 \left(p_1 - \sqrt{\frac{2\mu}{q_1} + h} \right) + \sigma_2(p_2 - C) \right],$$

with C the angular momentum constant and h the energy constant.

The canonical system (9) becomes :

$$q_1(t, \omega) = q_1(0, \omega) + \int_0^t p_1(s, \omega)\, \mathrm{d}s + \varepsilon \int_0^t \sigma_1(s, q_1(s, \omega), q_2(s, \omega))\, \mathrm{d}B_s(\omega),$$

$$q_2(t, \omega) = q_2(0, \omega) + \int_0^t \frac{p_2(s, \omega)}{q_1^2(s, \omega)}\, \mathrm{d}s + \varepsilon \int_0^t \sigma_2(s, q_1(s, \omega), q_2(s, \omega))\, \mathrm{d}B_s(\omega), \tag{14}$$

$$p_1(t, \omega) = p_1(0, \omega) - \int_0^t \left[\frac{\mu}{q_1^2(s, \omega)} + \frac{p_2^2(s, \omega)}{q_1^3(s, \omega)} \right] \mathrm{d}s - \varepsilon \int_0^t \frac{\mu}{q_1^2(s, \omega) \sqrt{\dfrac{2\mu}{q_1(s, \omega)} + h}}\, \sigma_1(s, q_1(s, \omega), q_2(s, \omega))\, \mathrm{d}B_s(\omega),$$

$$p_2(t, \omega) = p_2(0, \omega).$$

with the initial data

$$q_i(0, \omega) = q_i^0, \; i = 1, 2,$$

$$p_1(0, \omega) = \sqrt{\frac{2\mu}{q_1^0} + h}, \tag{15}$$

$$p_2(0, \omega) = C.$$

This system admits the solution :

$$\left\{q_1(t,\ \omega),\ q_2(t,\ \omega),\ \sqrt{\frac{2\mu}{q_1(t,\ \omega)}+h},\ C\right\} \tag{16}$$

where $\vec{q}(t,\ \omega)$ is the unique solution of the stochastic system :

$$\begin{aligned} q_1(t,\ \omega) &= q_1^0 + \\ &+\int_0^t \left[\sqrt{\frac{2\mu}{q_1(s,\ \omega)}+h}+\frac{\varepsilon^2}{2}\sum_{i=1}^{2}\frac{\partial\sigma_1}{\partial q_i}(s,\ \vec{q}(s,\ \omega))\cdot\sigma_i(s,\ \vec{q}(s,\ \omega))\right]\mathrm{d}s + \\ &+\varepsilon\int_0^t \sigma_1(s,\ \vec{q}(s,\ \omega))\,\mathrm{d}B_s(\omega), \\ q_2(t,\ \omega) &= q_2^0 + \\ &+\int_0^t \left[\frac{C}{q_1^2(s,\ \omega)}+\frac{\varepsilon^2}{2}\sum_{i=1}^{2}\frac{\partial\sigma_2}{\partial q_i}(s,\ \vec{q}(s,\ \omega))\cdot\sigma_i(s,\ \vec{q}(s,\ \omega))\right]\mathrm{d}s + \\ &+\varepsilon\int_0^t \sigma_2(s,\ \vec{q}(s,\ \omega))\,\mathrm{d}B_s(\omega). \end{aligned} \tag{17}$$

In order that the stochastic system (17) admits a solution, it is sufficient that the integrands to be *Lipschitz* continuous in both variables and with at most polynomial growth in the second variable (see [3] *Guikhman*, *Skorohod*, 1980). To accomplish this condition, ε must be chosen sufficiently small.

One can observe that, under the assumptions that the system (17) admits a unique solution, the component p_2 of the stochastic impulse still remains constant, with the same constant as in the classical situation. The natural further step from here is to study the qualitative behavior of the solution (16).

ACKNOWLEDGEMENTS. The authors are indebted to dr. Vasile Mioc for encouraging the elaboration of this paper.

REFERENCES

1. Arnold, V. : 1976, *Méthodes Mathématiques de la Mécanique Classique*, Moscou, editions Mir.
2. Bismut, J. M. : 1981, *Mécanique Aléatoire*, Lectures Notes in Math., 866, Springer-Verlag, Berlin, pag. 364–370.
3. Guikhman, I., Skorohod, A. : 1980, *Introduction à la Théorie des Processus Aléatoires*, Moscou, editions Mir, page 284, page 461.

Received on November 23, 1993