TWO-BODY PROBLEM WITH ANISOTROPIC GRAVITATIONAL CONSTANT

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Abstract. The two-body problem corresponding to a gravitational force with anisotropic G is treated by means of perturbation theory. The integration of Newton-Euler equations shows that the perturbed orbit is of elliptic type, lies in a fixed plane, and its semimajor axis, eccentricity, and argument of pericentre, obtained as functions of the argument of latitude, undergo 2π-periodic variations. The nodal period, determined with an accuracy of first order in a small parameter σ and third order in eccentricity, results to be shorter than the corresponding Keplerian period. The problem approached here constitutes a particular case of the very general two-body problem with changing equivalent gravitational parameter.

Key Words: celestial mechanics — equivalent gravitational parameter — anisotropy of gravitation — orbital motion — perturbation theory.

1. INTRODUCTION

Consider a point mass m orbiting a point mass M at distance r under the only action of the gravitational force

$$\mathbf{F} = -(GMm/r^3)\mathbf{r}. \tag{1}$$

An eventual anisotropy in the gravitational constant G:

$$G = G_\infty(1 + \varepsilon(\mathbf{V} \cdot \mathbf{r}/r)^2/c^2) \tag{2}$$

and the possibility to detect it by laboratory experiments have been discussed by Will (1971). In (2) G_∞ = Newtonian attraction constant, c = speed of light, $\varepsilon < 0.015$ (see Vinti, 1972), while $\mathbf{V}$ stands for the velocity of the laboratory system with respect to a so-called parametrized post-Newtonian (PPN) system. The quoted papers identify practically $\mathbf{V}$ with Sun's velocity relative to the Galaxy's centre.

The two-body problem which results with the force (1) in which G has the expression (2) was solved by Vinti (1972) on the basis of a Binet-type equation. He constructed a fixed ellipse (which is not an osculating one, and represents the solution of the homogeneous equation) to define orbital elements, and compared the motion corresponding to the general solution of the unhomogeneous Binet-type equation with the Keplerian motion obtained for $\varepsilon = 0$, especially as regards dynamic orbital parameters (true longitude, anomalies).

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Here we shall approach this problem in a perturbative manner. A comparison with Vinti's (1972) method and results will be made in the last section of this paper.

We have to emphasize the fact that this problem constitutes a particular case of a much more general one: the two-body problem with changing equivalent gravitational parameter (see Mioc et al., 1988 a).

2. EQUIVALENT GRAVITATIONAL PARAMETER

The force (1) being central, it is clear that the motion of m relative to M will be planar. Denote: β = angle between $\mathbf{V}$ and orbit plane, $\mathbf{V}_2$ = vector resulting by projecting $\mathbf{V}$ onto orbit plane, θ = angle between $\mathbf{V}_2$ and $\mathbf{r}$, w = angle from $\mathbf{V}_2$ to the ascending node of the orbit (defined with respect to a fundamental plane). It results immediately

$$\theta = w + u, \tag{3}$$

where u = argument of latitude. Notice that $\mathbf{V}_2$ is fixed in an inertial space (Vinti, 1972), and the orbit plane is fixed, too; hence both $\mathbf{V}_2$ and the line of nodes keep their positions constant in the orbit plane; thus w is constant.

Vinti's (1972) geometric considerations lead to

$$G = G_\infty(1 + \sigma \cos^2 \theta), \tag{4}$$

with $\sigma = \varepsilon (V/c)^2 \cos^2\beta$. Taking into account (1) – (4), the relative orbit of m will be described by the equation

$$\mathrm{d}^2r/\mathrm{d}t^2 - h^2/r^3 = -\mu/r^2, \tag{5}$$

in which h = constant angular momentum, and μ has the expression

$$\mu = H + JA^2 + KB^2 - LAB, \tag{6}$$

where we denoted:

$$H = G_\infty(M + m); \tag{7}$$

$$J = \sigma H \cos^2 w, \quad K = \sigma H \sin^2 w, \quad L = \sigma H \sin(2w); \tag{8}$$

$$A = \cos u, \quad B = \sin u. \tag{9}$$

One sees by (5) and (6) that this is a two-body problem in which the equivalent gravitational parameter μ (see Mioc et al., 1988 a) is variable. We shall treat this problem perturbatively; in other words, taking into account (5) and (6), we shall consider that in the right-hand side of (5) the term $-H/r^2$ features the effective Newtonian attraction, while the term $(LAB - JA^2 - KB^2)/r^2$ features a perturbing force due to the anisotropy of G. That is why we use hereafter a perturbation theory terminology.

3. EVOLUTION OF THE ORBITAL ELEMENTS

Since the perturbing force depends explicitly on u, we study the motion on the basis of Newton-Euler equations written with respect to the argument of latitude (e.g. Mioc and Radu, 1991; Mioc et al., 1992):

$$\begin{aligned}
&\mathrm{d}p/\mathrm{d}u = 2(Z/H)r^3T,\\
&\mathrm{d}\Omega/\mathrm{d}u = (Z/H)r^3W/(pD),\\
&\mathrm{d}i/\mathrm{d}u = (Z/H)r^3AW/p,\\
&\mathrm{d}q/\mathrm{d}u = (Z/H)\,(r^3kBCW/(pD) + r^2T(r(q+A)/p + A) + r^2BS),\\
&\mathrm{d}k/\mathrm{d}u = (Z/H)\,(-\,r^3qBCW/(pD) + r^2T(r(k+B)/p + B) - r^2AS),\\
&\mathrm{d}t/\mathrm{d}u = Zr^2(Hp)^{-1/2},
\end{aligned}\tag{10}$$

in which $Z = (1 - r^2C\dot{\Omega}/(Hp)^{1/2})^{-1}$, p = semilatus rectum, ω = argument of pericentre, e = eccentricity ($q = e\cos\omega$, $k = e\sin\omega$), Ω = longitude of ascending node, i = inclination ($C = \cos i$, $D = \sin i$), while S, T, W = radial, transverse, and binormal components of the perturbing acceleration, respectively.

But the perturbing force is radial; so

$$S = (IAB - JA^2 - KB^2)/r^2, \quad T = 0, \quad W = 0, \tag{11}$$

and the first three equations (10) yield immediately

$$p = p_0, \quad \Omega = \Omega_0, \quad i = i_0, \tag{12}$$

where subscripts refer to the initial position $u = u_0$. Hence the orbit plane is fixed (this is natural since the force acting on m is central), and the semilatus rectum keeps a constant value.

By (12), $Z = 1$; with (11) the fourth and fifth equations (10) acquire the form

$$\mathrm{d}q/\mathrm{d}u = (IAB^2 - JA^2B - KB^3)/H, \tag{13}$$

$$\mathrm{d}k/\mathrm{d}u = (JA^3 + KAB^2 - IA^2B)/H. \tag{14}$$

Observe that equations (13) and (14) are no more coupled. Integrating them, the behaviour of q and k in the interval $[u_0, u]$ is given respectively by

$$q = q_0 + (Q - Q_0)/(3H), \tag{15}$$

$$k = k_0 + (P - P_0)/(3H), \tag{16}$$

where we used the abbreviations

$$Q = Q(u) = LB^3 + (J - K)A^3 + 3KA, \tag{17}$$

$$P = P(u) = LA^3 - (J - K)B^3 + 3JB, \tag{18}$$

and $Q_0 = Q(u_0)$, $P_0 = P(u_0)$. It is easy to see that the expressions (15) and (16) are 2π-periodic functions of u.

Let us now pass to more intuitive orbital elements. Using the definitions of q and k, and expressions (15) — (18), then substituting (8) and (9) into the results, we get (to first order in σ) the behaviour of the eccentricity :

$$\begin{aligned} e = e_0 - \sigma((1/2)\sin((u + u_0)/2 - 2w + \omega_0)\sin((u - u_0)/2) + \\ + (1/6)\sin(3(u + u_0)/2 + 2w - \omega_0)\sin(3(u - u_0)/2) - \\ - \cos(2w)\sin((u + u_0)/2 + \omega_0)\sin((u - u_0)/2) + \\ + \sin((u + u_0)/2 - \omega_0)\sin((u - u_0)/2)), \end{aligned} \tag{19}$$

and that of the argument of pericentre :

$$\begin{aligned} \omega = \omega_0 - (\sigma/e_0)((1/2)\cos((u + u_0)/2 - 2w + \omega_0)\sin((u - u_0)/2) - \\ - (1/6)\cos(3(u + u_0)/2 + 2w - \omega_0)\sin(3(u - u_0)/2) - \\ - \cos(2w)\cos((u + u_0)/2 + \omega_0)\sin((u - u_0)/2) - \\ - \cos((u + u_0)/2 - \omega_0)\sin((u - u_0)/2)). \end{aligned} \tag{20}$$

With $p = a(1 - e^2)$, (12), and (19), we easily obtain the evolution of the semimajor axis a :

$$\begin{aligned} a = a_0 - 2a_0e_0\sigma((1/2)\sin((u + u_0)/2 - \\ - 2w + \omega_0)\sin((u - u_0)/2) + \\ + (1/6)\sin(3(u + u_0)/2 + 2w - \omega_0)\sin(3(u - u_0)/2) - \\ - \cos(2w)\sin((u + u_0)/2 + \omega_0)\sin((u - u_0)/2) + \\ + \sin((u + u_0)/2 - \omega_0)\sin((u - u_0)/2))/(1 - e_0^2). \end{aligned} \tag{21}$$

It is clear that expressions (19), (20), and (21) are 2π-periodic functions of u, too.

By the last two expressions (12) and (19)—(21) we can say that along the interval $[u_0, u_0+2\pi]$ the osculating orbit lies in a fixed plane, but undergoes changes, in shape, dimensions, and orientation. As to the evolution of these changes, putting $\Delta y = y(u) - y_0$, with $y \in \{a, e, \omega\}$, and knowing that $y(u_0) = y(u_0 + 2\pi) = y_0$, there are the following possible situations:

(i) Δy does not admit zeros in the interval $(u_0, u_0 + 2\pi)$, that is, during a revolution (except, of course, the starting and ending points) the respective element is systematically greater or smaller than its initial value;

ii) Δy admits zeros in the interval $(u_0, u_0 + 2\pi)$, that is, during a revolution the respective element oscillates around its initial value.

The kind of change undergone by the element y depends on the values assigned to the initial parameters u_0 and ω_0, and to the constant w.

For the trajectory of the point mass m, using the orbit equation in polar coordinates $r = p / (1 + e \cos v)$, where v = true anomaly, under the form

$$r = p/(1 + qA + kB), \tag{22}$$

and taking into account (12), (15), and (16), we get

$$p_0/r = 1 + (q_0 + (Q - Q_0)/(3H))A + (k_0 + (P - P_0)/(3H))B, \tag{23}$$

with Q, P (and Q_0, P_0) provided by (17), (18), respectively. We can hence say that equation (23) describes an ellipse whose eccentricity undergoes continuous and 2π-periodic changes, according to one of the above mentioned situations.

4. THE NODAL PERIOD

The nodal period, defined by

$$T_\Omega = \int_0^{2\pi} (\mathrm{d}t/\mathrm{d}u)\,\mathrm{d}u, \tag{24}$$

will be estimated by means of the method proposed by Zhongolovich (1960) and extended by Mioc (1992). According to this method, whose principles were also sketched in the papers of Mioc (1980), Mioc and Blaga (1991), Mioc and Radu (1991 b), Mioc et al. (1992) and will not be repeated here, the nodal period (perturbed by an arbitrary force depending on a small parameter σ) is given to first order in σ by

$$T_\Omega = T_0 + \Delta^{(1)} T_\Omega, \tag{25}$$

where

$$T_0 = p_0^{3/2} H^{-1/2} \int_0^{2\pi} g^{-2}\,\mathrm{d}u, \tag{26}$$

and

$$\Delta^{(1)}T_\Omega = p_0^{3/2}H^{-1/2}(-2I_q - 2I_k + (3/2)p_0^{-1}I_p + p_0^2H^{-1}I_\sigma), \quad (27)$$

with

$$I_p = \int_0^{2\pi} g^{-2}\,\Delta p\,\mathrm{d}u,\quad I_q = \int_0^{2\pi} g^{-3}A\Delta q\,\mathrm{d}u,$$

$$I_k = \int_0^{2\pi} g^{-3}B\,\Delta k\,\mathrm{d}u,\quad I_\sigma = \int_0^{2\pi} g^{-5}B(\partial(CW/D)/\partial\sigma)_0\sigma\,\mathrm{d}u. \quad (28)$$

In these formulae H is given by (7), $\Delta y = y(u) - y_0$, $y \in \{p, q, k\}$, while $g = g(u) = 1 + q_0A + k_0B$.

In our problem, where the small parameter is just σ which appears in (4), we observe that $I_p = 0$, $I_\sigma = 0$, expand g^{-2} and g^{-3} to third order in q_0, k_0 (or equivalently in e_0), replace the results and Δq, Δk given by (15) — (16) into (26) and (28), perform the integrations, and calculate the expression (27). The results are:

$$T_0 = 2\pi p_0^{3/2}H^{-1/2}(1 + 3(q_0^2 + k_0^2)/2), \quad (29)$$

$$\Delta^{(1)}T_\Omega = -\pi p_0^{3/2}H^{-3/2}(2(J+K) + 2(q_0Q_0 + k_0P_0) + $$
$$+ 2q_0k_0L + 5(q_0^2J + k_0^2K) + 7(q_0^2K + k_0^2J) + \quad (30)$$
$$+ 5(q_0^2 + k_0^2)(q_0Q_0 + k_0P_0)).$$

Now, in order to compare the perturbed nodal period with the corresponding Keplerian period, we replace in (29) — (30): the definitions of q and k, the relation $p = a(1 - e^2)$, the expressions (17) — (18), then (8) and (9); performing the sum (25), we get the perturbed nodal period with an accuracy of first order in σ and third order in eccentricity:

$$T_\Omega = 2\pi a_0^{3/2}H^{-1/2}(1 - \sigma f(e_0, \omega_0, u_0, w)), \quad (31)$$

where

$$f = 1 + [\cos\omega_0(2\sin w\sin^2 u_0\sin(w + u_0) + \cos u_0(\sin^2 w + \cos^2 u_0)) + $$
$$+ \sin\omega_0(2\cos w\cos^2 u_0\sin(w + u_0) + \sin u_0(\cos^2 w + \sin^2 u_0))]\times \quad (32)$$
$$\times(e_0 + e_0^3) + (1 + \sin^2(w + \omega_0))e_0^2.$$

As long as f has positive values (and this is generally the case, as we shall see), the nodal period T_Ω will be shorter than the corresponding Keplerian period $T_K = 2\pi a_0^{3/2}H^{-1/2}$, the difference being of order at most

σT_K. So, the anisotropy in G has the tendency to accelerate the motion. This speeding is maximum when the initial position is in pericentre σ $(u_0 = \omega_0)$, and in addition the initial line of nodes is perpendicular to $\mathbf{V}_2(w + u_0 = \pi/2$ or $3\pi/2)$, that is, (32) reduces to

$$f = 1 + 2e_0 + 2e_0^2 + 2e_0^3, \tag{33}$$

and minimum when the initial position is in apocentre $(u_0 = \omega_0 + \pi)$, and the above second condition is fulfilled, too; in this case (32) reduces to

$$f = 1 - 2e_0 + 2e_0^2 - 2e_0^3. \tag{34}$$

Two remarks are to be made. Firstly, observe from (31) and (34) that to have $T_\Omega > T_K$ (decelerated motion) the initial eccentricity must exceed 0.65 roughly. But for such high eccentric initial orbits an accuracy of third order in e_0 is absolutely insufficient; therefore, with the accuracy of (31) – (32), we can say that the nodal period is always shorter than the corresponding Keplerian period. Secondly, observe from (3) and (4) that the second condition imposed to u_0 in order to get an extremum for f leads to an initial value of G equal to G_∞ (initially Keplerian motion governed by the purely Newtonian attraction; this is a natural and usual condition).

5. CONCLUSIONS AND COMMENTS

The possible anisotropy in G presumed by Will (1971) to be of the form (2) leads to a two-body problem in which the gravitational force is anisotropic. Vinti (1972), whose paper constituted the departure point for our investigation, solved this problem by integrating a Binet-type equation, while our treatment is a perturbative one based on the integration of Newton-Euler equations. More concretely, Vinti compared the motion described by his equation (through orbital elements defined on a non-osculating fixed ellipse) with the unperturbed motion obtained by making $\varepsilon = 0$ in (2); we consider the anisotropy of G to be a perturbing factor and compare the real perturbed orbit with the initial Keplerian orbit.

The major purpose of Vinti's (1972) paper is to estimate quantitatively the effects of the anisotropy of G on orbits in the solar system (by identifying practically $\mathbf{V}$ with Sun's velocity with respect to Galaxy's centre), The present paper studies the evolution of an arbitrary elliptic-type orbit in the case of an anisotropic gravitational constant.

Our results indicate an elliptic-type motion in a fixed plane. The semimajor axis, eccentricity, and argument of pericentre, obtained as functions of u, present 2π-periodic variations. The nodal period, determined with a first order accuracy in σ and third order accuracy in e_0, is shorter than the corresponding Keplerian period, the difference being of order at most σT_K. A speeding of the motion having this order of magnitude results from Vinti's investigation, too.

The use of the argument of latitude as independent variable (and the consecutive determination of the nodal period as basic time interval) allows the study of very low eccentric orbits, even circular.

By (5) and (6) this is a problem which belongs — we repeat — to the general two-body problem with changing equivalent gravitational parameter in the meaning of Mioc et al. (1988 a). The variation of μ can be of very different natures (see e.g. Duboshin, 1963; Savedoff and Vila, 1964; Glikman, 1976, 1978; Saslaw, 1978; Giurgiu, 1988) and of various types. Among such changes, those with cyclic character, especially the periodic ones, were approached by Saslaw (1978), Mioc et al. (1988 b, c), Mioc (1989), Şelaru et al. (1992). The anisotropy of μ constitutes an apart case of periodic variation (see Saslaw, 1978), and this class of problems includes the situation we studied in this paper, too.

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