DOI 10.59277/RoAJ.2025.1-2.05

THE COLLINEAR RESTRICTED 4-BODY PROBLEM WITH LOGARITHMIC POTENTIAL

ANNA-MONIKA MUSCAȘ

The Center for Doctoral University Studies, University of Oradea
University Street 1, 410087 Oradea, Romania
Email*: monicaszilagyi@gmail.com*

Abstract. In this paper we study dynamical aspects of the collinear restricted 4-body problem with logarithmic potential. We determine equilibria, their stability, and prove the existence of some periodic orbits using the Lyapunov Center Theorem. We also prove that some dynamical features continue from the restricted to the non-restricted problem.

Key words: collinear restricted 4-body problem, logarithmic potential, equilibria, stability, periodic orbits, continuation of the dynamics from restricted to non-restricted problems.

1. INTRODUCTION

In a fictitious two dimensional universe, the gravitational potential between two mass points m_1 and m_2 is $U(\mathbf{r}) = Gm_1m_2 \ln|\mathbf{r}|$ where $\mathbf{r}$ is the vector between the two mass points and G is a constant. This follows from the Laplace equation in $\mathbb{R}^2$ and by considering the potential as being directly proportional to the product of the masses, similarly as in the Newtonian case. Logarithm potentials are used in astrophysics to construct models of galaxies that are self-consistent; for example, see Binney *et al.* (1987); Miralda-Escude *et al.* (1989), and more recently, Belmonte *et al.* (2007); Valluri *et al.* (2012). An attractive law, the logarithm "pull" is weaker at close range than in any law of the form $-1/r^{\alpha}$, $\alpha > 0$, but stronger at long range. In fact, in the logarithm n-body problem all orbits are bounded, regardless of the energy level.

Previous work on the various n-body problems with logarithm interaction is found in Stoica *et al.* (2003); Vidal (2007); Castelli *et al.* (2011); Saha *et al.* (2023); Muscas *et al.* (2024, 2025). Numerical simulations for large n in an astrophysical context are presented in Binney *et al.* (1987); Miralda-Escude *et al.* (1989); Belmonte *et al.* (2007); Valluri *et al.* (2012).

A generalization of the restricted 3-body problem, the restricted 4-body problem describes the motion of an infinitesimal mass under the attraction of 3 primaries situated on a line that rotates with constant angular velocity about its centre. In this paper we investigate dynamical aspects of restricted 4-body problem with logarithmic potential in the case of equal-mass primaries with a collinear configuration.

Modelled in a rotating coordinate system moving along with the primaries, the 4-restricted problem benefits of the conservation of energy expressed as the Jacobi integral. We determine the zero-velocity curves and depict the Hill'd regions of motion. Next we determine the equilibria and their stability. We find six equilibria, four aligned with the primaries (two in exteriors and two in interior) and two forming isosceles triangles (see Figure 1). Direct calculations of the linearisation spectrum at equilibria proves that the collinear ones equilibria are unstable, whereas those of isosceles configuration are linearly stable. This is in agreement with the more general findings of Muscas *et al.* (2025). Next, by applying the Lyapunov Centre Theorem Abraham *et al.* (1987), we prove the existence of one-parameter families of periodic orbits emanating from the unstable equilibria. Further, we determine the linear expression of the periodic orbits bifurcating from the equilibrium points. In the last Section we apply a continuation theorem and show that all equilibria and elementary periodic orbits can be continued in the (non-restricted) 4-body problem with logarithm potential.

The paper is organised as follows: in Section 2 we set up the problem. In Section 3 we determine the equilibria. In the next Section we find the Jacobi integral and the associated zero velocity, and discuss the Hill's regions of motion. Section 5 investigates the equilibria stability. In Section 6 we apply the Lyapunov Centre Theorem to determine the existence of periodic orbits near equilibria, find the linear expression of these periodic orbits, and apply the continuation results from the restricted to the full problem.

2. EQUATIONS OF MOTION

Taking the unit of mass equal to the masses of the three primaries and since a central configuration is invariant under rotations and homotheties through its center of mass without loss of generality we can assume that the position vectors q_i's for $i = 1,2,3$ of the three primaries with masses $m_1 = m_2 = m_3 = 1$ are

$$q_1 = (q_{11}, q_{12}) = (-1, 0),\ q_2 = (q_{21}, q_{22}) = (0, 0),\ q_3 = (q_{31}, q_{32}) = (1, 0). \tag{1}$$

We denote the position of the infinitesimal mass $m_4 = 0$ by $q_4 = (q_{41}, q_{42}) = (x, y)$.

For $i = 1,2,3$, let $p_i = (p_{i1}, p_{i2})$ be the momentum vectors for the three primaries in a fixed coordinates frame. We introduce a set of coordinates that uniformly rotate with constant frequency ω by

$$u_i = \exp(\omega K t) q_i,\ \ v_i = \exp(\omega K t) p_i,\ i = 1,2,3,$$

where

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \exp(\omega K t) = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix},$$

and $u_i = (u_{i1}, u_{i2}), v_i = (v_{i1}, v_{i2})$ for $i = 1,2,3$. The first objective is to determine the frequency ω of the rotation system of coordinates in such a way that the three primaries are in a circular periodic orbit of the 3-body problem, in other words that in a rotating coordinate system with frequency ω the three primaries remain at rest, or equivalently find ω in order that the positions $(1,0)$, $(0,0)$ and $(-1,0)$ of the primaries be a critical point for the Hamiltonian system with the Hamiltonian

$$\begin{aligned} H_3 = & \sum_{i=1}^{3} \frac{\|v_i\|^2}{2} - \omega \sum_{i=1}^{3} u_i^T K v_i + \ln\|u_1 - u_2\| + \ln\|u_1 - u_3\| + \ln\|u_2 - u_3\|, \\ = & \sum_{i=1}^{3} \frac{\|v_i\|^2}{2} - \omega \sum_{i=1}^{3} u_i^T K v_i - U(u_1, u_2, u_3), \end{aligned}$$

of the 3-body problem in the rotating coordinates.

The equations of motion are

$$\dot{u}_i = v_i + \omega K u_i, \qquad \dot{v}_i = \omega K v_i + \frac{\partial U}{\partial u_i}. \tag{2}$$

Taking into account that we want $u_1 = (-1,0), u_2 = (0,0), u_3 = (1,0)$ be a solution for (2), then we obtain that $v_1 = (0,-\omega)$, $v_2 = (0,0)$, $v_3 = (0,\omega)$ and $\omega = \sqrt{3/2}$.

The Hamiltonian of the infinitesimal mass in the rotating system of coordinates for the circular restricted collinear 4-body problem is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \omega(p_x y - p_y x) + \ln r_{14} + \ln r_{24} + \ln r_{34}, \tag{3}$$

where

$$r_{14}^2 = (x+1)^2 + y^2, \; r_{24}^2 = x^2 + y^2, \; r_{34}^2 = (x-1)^2 + y^2.$$

For more details on the obtation of this Hamiltonian see Meyer *et al.* (2009); Szebehely (1967); Valtonen *et al.* (2005).

The equations of motion of the infinitesimal mass are

$$\begin{aligned} \dot{x} &= p_x + \omega y, \\ \dot{y} &= p_y - \omega x, \\ \dot{p}_x &= -\frac{\partial}{\partial x}(\ln r_{14} + \ln r_{24} + \ln r_{34}) + \omega p_y, \\ \dot{p}_y &= -\frac{\partial}{\partial y}(\ln r_{14} + \ln r_{24} + \ln r_{34}) - \omega p_x. \end{aligned} \tag{4}$$

The Newtonian equations of motion are obtained by eliminating the momenta from equations (4)

$$\ddot{x}-2\omega\dot{y}=\frac{\partial\Omega}{\partial x},\quad \ddot{y}+2\omega\dot{x}=\frac{\partial\Omega}{\partial y},\tag{5}$$

where

$$\Omega=\frac{1}{2}\omega^2(x^2+y^2)-\ln r_{14}-\ln r_{24}-\ln r_{34},$$

can be considered an effective potential.

3. EQUILIBRIUM POINTS

The equations of motion (5) have a trivial solution $\dot{x}=\dot{y}=0$ if the derivatives of Ω vanish. Thus the infinitesimal body remain at rest at such points, the equilibrium points. We now compute them. Calculating the derivatives of Ω and equal them to zero we get the following pair of equations for the equilibrium points points

$$\begin{aligned}\frac{\partial\Omega}{\partial x}&=\omega^2x-\frac{x+1}{r_{14}^2}-\frac{x}{r_{24}^2}-\frac{x-1}{r_{34}^2}=0,\\ \frac{\partial\Omega}{\partial y}&=y\left(\omega^2-\frac{1}{r_{14}^2}-\frac{1}{r_{24}^2}-\frac{1}{r_{34}^2}\right)=0.\end{aligned}\tag{6}$$

From the second equation we have two cases.

The first case is when $y=0$. Substituting this into the first equation we get:

$$\frac{3}{2}x-\frac{1}{x+1}-\frac{1}{x}-\frac{1}{x-1}=0.\tag{7}$$

The roots are

$$x_{1,2}=\pm\sqrt{\frac{9-\sqrt{57}}{6}},\ \ x_{3,4}=\pm\sqrt{\frac{9+\sqrt{57}}{6}}.\tag{8}$$

The second case $y\neq 0$ reduces to the equation

$$\omega^2-\frac{1}{r_{14}^2}-\frac{1}{r_{24}^2}-\frac{1}{r_{34}^2}=0.\tag{9}$$

Multiplying this equation by $-x$ and adding it to the first equation of (6) we obtain $x=0$. Then from (9) we have the following equations for y

$$3y^4-3y^2-2=0,\ \text{with real roots } y_{5,6}=\pm\sqrt{\frac{3+\sqrt{33}}{6}}.$$

In summary we have obtained six equilibrium points $L_i=(x_i,0)$ for $i=1,2,3,4$, and $L_j=(0,y_j)$ for $j=5,6$.

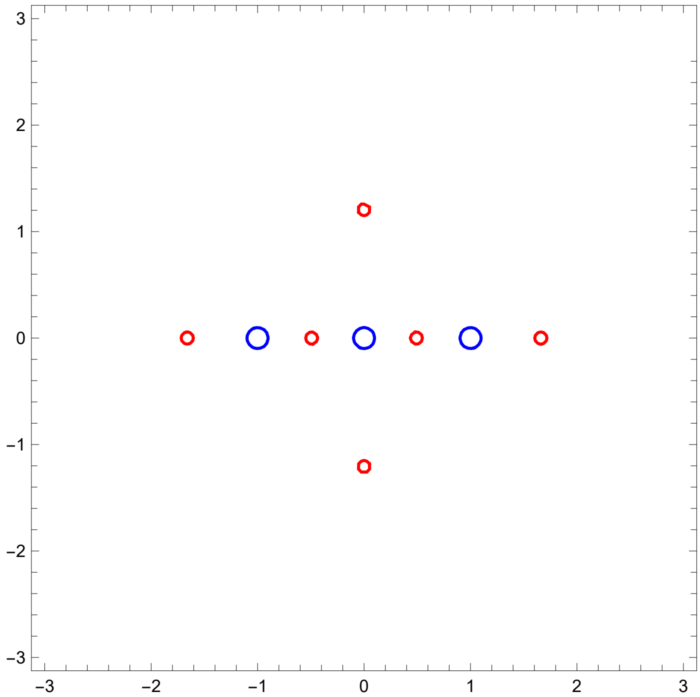


Fig. 1 – The red circles are the equilibrium points and the blue ones are the primaries.

If we calculate the Hessian of the function Ω at these equilibria we get

$$H(L_1)=H(L_4)=H\Big(\pm\sqrt{\frac{9+\sqrt{57}}{6}},0\Big)=\begin{pmatrix} 4.293812086773471.. & 0\\ 0 & -1.2938120867734706.. \end{pmatrix},$$

so L_1 and L_4 are saddles of Ω;

$$H(L_2)=H(L_3)=H\Big(\pm\sqrt{\frac{9-\sqrt{57}}{6}},0\Big)=\begin{pmatrix} 9.956187913226529.. & 0\\ 0 & -6.95618791322653.. \end{pmatrix},$$

so L_2 and L_3 are also saddles of Ω;

$$H(L_5)=H(L_6)=H\Big(0,\pm\sqrt{\frac{3+\sqrt{33}}{6}}\Big)=\begin{pmatrix} 0.6623670226447177.. & 0\\ 0 & 2.337632977355282.. \end{pmatrix},$$

so L_5 and L_6 are minima of Ω.

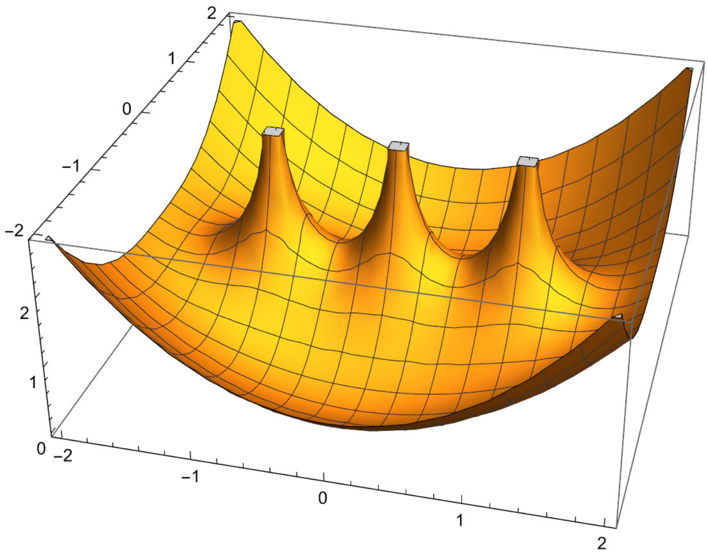


Fig. 2 – Plot of the surface $z = \Omega(x, y)$.

4. JACOBIAN INTEGRAL AND HILL'S REGIONS

We multiply the equations of motion (5) by $\dot{x}$ and $\dot{y}$ and adding them we obtain

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\frac{\partial \Omega}{\partial x} + \dot{y}\frac{\partial \Omega}{\partial y}.$$

Since the function Ω does not depend on the time, on the right hand side of the previous equality we have the total time derivative of Ω, so we have that

$$\frac{d}{dt}(\dot{x}^2 + \dot{y}^2) = 2\frac{d\Omega}{dt}.$$

Integrating it with respect to the time t we get the Jacobian first integral

$$C = 2\Omega - v^2,$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the velocity of the infinitesimal mass, and C is a constant, called the Jacobian constant.

Since the square of the velocity cannot be negative we have that

$$\Omega(x,y) = \frac{3}{4}(x^2 + y^2) + \frac{1}{\sqrt{(x+1)^2 + y^2}} + \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{(x-1)^2 + y^2}} \geq \frac{C}{2}.$$

Now we study the curves of zero velocity. If $v = 0$ the Jacobian first integral reduces to $2\Omega = C$. When the Jacobi constant varies we obtain different curves of zero velocity. These curves of zero velocity are the boundaries of the Hill's regions, *i.e.*the set of points (x, y) of the plane where $\Omega(x, y) \geq C/2$, *i.e.*the regions of the

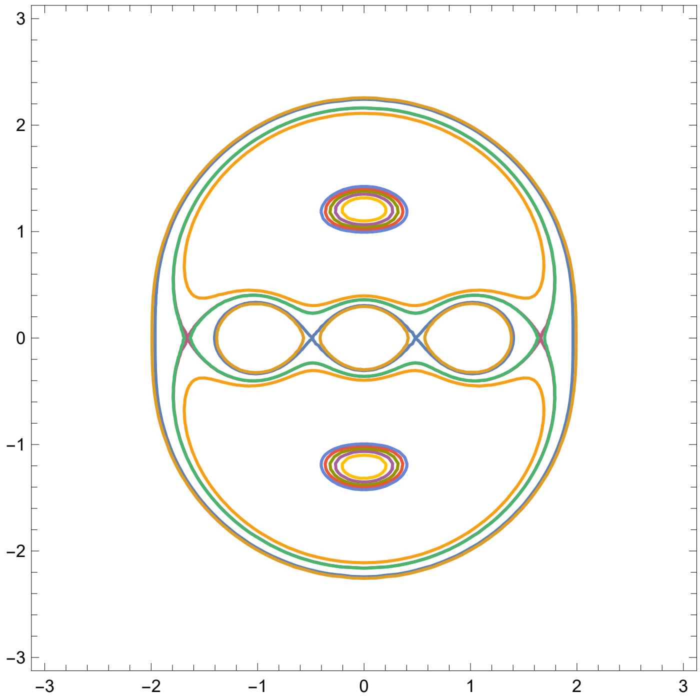


Fig. 3 – The curves of zero velocity for different values of C.

plane where the motion is possible for the value C of the Jacobi constant. This means that the motion cannot take place in the regions where $\Omega(x,y) < C/2$.

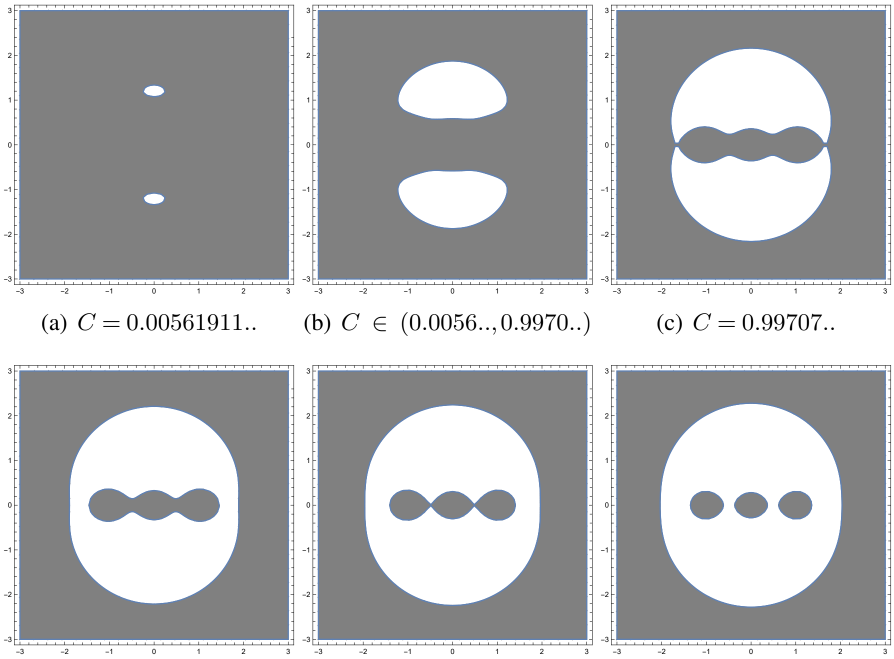
If we compute the values of the function Ω at the equilibrium points we get

$$\Omega\Big(\pm\sqrt{\tfrac{9+\sqrt{57}}{6}},0\Big) = 0.9970702928603083..,$$

$$\Omega\Big(\pm\sqrt{\tfrac{9-\sqrt{57}}{6}},0\Big) = 1.1679801887419932..,$$

$$\Omega\Big(0,\pm\sqrt{\tfrac{3+\sqrt{33}}{6}}\Big) = 0.005619106274313834...$$

This means that no level curves exist when $\Omega < 0.005619106274313834..$, and consequently no motion exists for the values of $\Omega < 0.005619106274313834...$ The equilibrium points are located at critical values where the curves of zero velocity change their topology. In Figure 4 the Hill's regions where the motion is allowed are the white regions.



(a) $C = 0.00561911..$ (b) $C \in (0.0056.., 0.9970..)$ (c) $C = 0.99707..$

(d) $C \in (0.9970.., 1.1679..)$ (e) $C = 1.16798..$ (f) $C > 1.16798..$

Fig. 4 – The curves of zero velocity and their Hill's regions.

5. STABILITY OF THE EQUILIBRIUM POINTS

At the equilibrium points the linearized system corresponding to system (4) has the matrix

$$\begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \dfrac{\partial^2\Omega}{\partial x^2} - \omega^2 & \dfrac{\partial^2\Omega}{\partial x \partial y} & 0 & \omega \\ \dfrac{\partial^2\Omega}{\partial x \partial y} & \dfrac{\partial^2\Omega}{\partial y^2} - \omega^2 & -\omega & 0 \end{pmatrix}. \tag{10}$$

For the equilibrium points $L_i = (x_i, 0)$ for $i = 1, 2, 3, 4$ on the x-axis, the above matrix becomes

$$\begin{pmatrix} 0 & \sqrt{3/2} & 1 & 0 \\ -\sqrt{3/2} & 0 & 0 & 1 \\ \alpha_i & 0 & 0 & \sqrt{3/2} \\ 0 & -\alpha_i & -\sqrt{3/2} & 0 \end{pmatrix},$$

where

$$\alpha_i = \frac{1}{(x_i+1)^2} + \frac{1}{x_i^2} + \frac{1}{(x_i-1)^2}, \quad \text{for } i = 1,2,3,4.$$

The eigenvalues of this matrix at the equilibrium points L_1 and L_4 are

$$\pm\frac{1}{2}i\sqrt{\frac{3}{2}(19-\sqrt{57})}, \quad \pm\frac{1}{4}(3-\sqrt{57}),$$

and the eigenvalues at the equilibrium points L_2 and L_3 are

$$\pm\frac{1}{2}i\sqrt{\frac{3}{2}(19+\sqrt{57})}, \quad \pm\frac{1}{4}(3+\sqrt{57}).$$

Since for each one of these equilibrium points one eigenvalue is real positive, all these equilibrium points are unstable.

For the equilibrium points $L_j = (0, y_j)$ for $j = 5,6$ on the y-axis, the matrix (10) becomes

$$\begin{pmatrix} 0 & \sqrt{3/2} & 1 & 0 \\ -\sqrt{3/2} & 0 & 0 & 1 \\ \beta_i & 0 & 0 & \sqrt{3/2} \\ 0 & -\beta_i & -\sqrt{3/2} & 0 \end{pmatrix},$$

where

$$\beta_i = -\frac{1+3y^4}{y^2(1+y^2)^2}, \quad \text{for } i = 5,6,.$$

The eigenvalues of this matrix at the equilibrium points L_5 and L_6 are

$$\pm\frac{1}{2}i\sqrt{\frac{3}{2}(-11+3\sqrt{33})}, \qquad \pm\frac{1}{2}i\sqrt{\frac{3}{2}(19-3\sqrt{33})}.$$

Hence the equilibrium points L_5 and L_6 are stable.

In summary we have proved the next result.

Theorem 1. *The four equilibrium points situated on the x-axis are unstable and the equilibrium points situated on the y-axis are linearly stable.*

Remark 2. *We emphasize that for the Newtonian potential all six equilibrium points are unstable (see Llibre et al. (2021)).*

6. PERIODIC ORBITS

6.1. LYAPUNOV FAMILIES AT THE EQUILIBRIUM POINTS L_i, $i = 1,2,3,4,5,6$

Recall that a first integral F is called *nondegenerate* at a point x_0 if its gradient $\nabla F(x_0) \neq 0$. We note that the Hamiltonian (3) is always a nondegenerate first integral along any nonequilibrium solution.

Theorem 3 (Lyapunov Center Theorem)**.** *Assume that the differential system $\dot{x} = f(x)$ admits a nondegenerate integral and has an equilibrium point with exponents $\pm\omega i, \lambda_3, \ldots, \lambda_m$, where $i\omega \neq 0$ is pure imaginary. If $\lambda_j/i\omega$ is never an integer for $j = 3, \ldots, m$, then there exists an one-parameter family of periodic orbits emanating from the equilibrium point. Moreover, when approaching the equilibrium point along the family, the periods tend to $2\pi/\omega$ and the nontrivial multipliers tend to $\exp(2\pi\lambda_j/\omega)$ for $j = 3, \ldots, m$.*

For a proof of Theorem 3 see Theorem 9.2.1 of Meyer *et al.* (2009). We now apply the above to our particular problem.

Corollary 4. *In the collinear restricted 4-body problem with logarithm potential, at the equilibrium points L_1 (and L_4) we have a pair of purely imaginary eigenvalues $\pm 2.07215..i$ and a pair of real eigenvalues $\pm 1.13745..$, so the Lyapunov center theorem implies that there is a one-parameter family $\gamma_\varepsilon(t)$ of periodic orbits that approaches L_1 (and L_4) as $\varepsilon \to 0$, whose period approach $2\pi/2.07215..$ as $\varepsilon \to 0$.*

Further at the equilibrium points L_2 (and L_3) we have a pair of purely imaginary eigenvalues $\pm 3.15534..i$ and a pair of real eigenvalues $\pm 2.63745..$, so the Lyapunov center theorem implies that there is an one-parameter family $\gamma_\varepsilon(t)$ of periodic orbits that approaches L_2 (and L_3) as $\varepsilon \to 0$, whose period approach $2\pi/3.15534..$ as $\varepsilon \to 0$.

6.2. THE LINEAR EXPRESSION OF THE PERIODIC ORBITS BIFURCATING FROM THE EQUILIBRIUM POINTS L_i, $i = 1,2,3,4$

At these four equilibrium points we have $\Omega_{xx} > 0$, $\Omega_{xy} = 0$ and $\Omega_{yy} < 0$ and we denote by $\pm\lambda$ their two real eigenvalues, and by $\pm\beta i$ their two purely imaginary eigenvalues. The linearised equations of motion (5), called the linear variational equations, are

$$\ddot{x}-2\omega\dot{y}=x\frac{\partial^2\Omega}{\partial x^2}+y\frac{\partial^2\Omega}{\partial x\partial y},\quad \ddot{y}+2\omega\dot{x}=x\frac{\partial^2\Omega}{\partial x\partial y}+y\frac{\partial^2\Omega}{\partial y^2},\tag{11}$$

where the second order partial derivatives of Ω are calculated at the equilibrium points L_i for $i=1,2,3,4$. The solution of these linear variational equations is

$$\begin{aligned}x(t)&=A_1e^{\lambda t}+A_2e^{-\lambda t}+A_3e^{\beta ti}+A_4e^{-\beta ti},\\ y(t)&=B_1e^{\lambda t}+B_2e^{-\lambda t}+B_3e^{\beta ti}+B_4e^{-\beta ti}.\end{aligned}\tag{12}$$

The coefficients $A_1,\ldots,A_4$, $B_1,\ldots,B_4$ are not independent because from the first equation of (11) we have

$$\begin{aligned}B_1&=\frac{\lambda^2-\Omega_{xx}|_{L_1-L_4}}{2\omega\lambda}A_1=\alpha_1A_1,\\ B_2&=-\frac{\lambda^2-\Omega_{xx}|_{L_1-L_4}}{2\omega\lambda}A_2=-\alpha_1A_2,\\ B_3&=\frac{\beta^2+\Omega_{xx}|_{L_1-L_4}}{2\omega\beta}A_3i=\alpha_3A_3i,\\ B_4&=-\frac{\beta^2+\Omega_{xx}|_{L_1-L_4}}{2\omega\beta}A_4i=-\alpha_3A_4i,\end{aligned}\tag{13}$$

taking into account that $\Omega_{xy}=0$ at the equilibrium points L_i for $i=1,2,3,4$. As in Szebehely (1967) we can see that the four initial conditions of (11) will completely determine the coefficients $A_1,\ldots,A_4$, $B_1,\ldots,B_4$. Thus we have

$$\begin{aligned}x_0=x(t_0)&=A_1e^{\lambda t_0}+A_2e^{-\lambda t_0}+A_3e^{\beta t_0i}+A_4e^{-\beta t_0i},\\ \dot{x}_0=\dot{x}(t_0)&=\lambda A_1e^{\lambda t_0}-\lambda A_2e^{-\lambda t_0}+\beta A_3e^{\beta t_0i}i-\beta A_4e^{-\beta t_0i}i,\\ y_0=y(t_0)&=\alpha_1A_1e^{\lambda t_0}-\alpha_1A_2e^{-\lambda t_0}+\alpha_3A_3e^{\beta t_0i}i-\alpha_3A_4e^{-\beta t_0i}i,\\ \dot{y}_0=\dot{y}(t_0)&=\alpha_1\lambda A_1e^{\lambda t_0}+\alpha_1\lambda A_2e^{-\lambda t_0}-\alpha_3\beta A_3e^{\beta t_0i}-\alpha_3\beta A_4e^{-\beta t_0i}.\end{aligned}\tag{14}$$

Taking into account (13) the determinant of this system is

$$\begin{vmatrix}1&1&1&1\\ \lambda&-\lambda&\beta i&-\beta i\\ \alpha_1&-\alpha_1&\alpha_3i&-\alpha_3i\\ \alpha_1\lambda&\alpha_1\lambda&-\alpha_3\beta&-\alpha_3\beta\end{vmatrix}=\frac{\Omega_{xx}|_{L_1-L_4}(\lambda^2+\beta^2)^2}{\omega^2\lambda\beta}i\neq 0.$$

Therefore the solution of system (14) is

$$\begin{aligned}
A_1 &= \frac{e^{-\lambda t_0}}{2}\Big[\frac{\alpha_3\beta x_0}{\alpha_3\beta+\alpha_1\lambda}-\frac{\alpha_3\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3}+\frac{\beta y_0}{\alpha_1\beta-\lambda\alpha_3}+\frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big],\\
A_2 &= \frac{e^{\lambda t_0}}{2}\Big[\frac{\alpha_3\beta x_0}{\alpha_3\beta+\alpha_1\lambda}+\frac{\alpha_3\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3}-\frac{\beta y_0}{\alpha_1\beta-\lambda\alpha_3}+\frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big],\\
A_3 &= \frac{e^{-i\beta t_0}}{2}\Big[\frac{\alpha_1\lambda x_0}{\alpha_3\beta+\alpha_1\lambda}-i\frac{\alpha_1\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3}+i\frac{\lambda y_0}{\alpha_1\beta-\lambda\alpha_3}-\frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big],\\
A_4 &= \frac{e^{i\beta t_0}}{2}\Big[\frac{\alpha_1\lambda x_0}{\alpha_3\beta+\alpha_1\lambda}+i\frac{\alpha_1\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3}-i\frac{\lambda_1 y_0}{\alpha_1\beta-\lambda\alpha_3}-\frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big].
\end{aligned}\tag{15}$$

We are looking for a periodic solution, so we must have $A_1 = A_2 = 0$. If we choose the initial conditions in order that $A_1 = A_2 = 0$, then from the first two relations of (15) we get that, if x_0, y_0 are arbitrarily selected, then $\dot{x}_0 = \beta y_0/\alpha_3$ and $\dot{y}_0 = -\alpha_3\beta x_0$. Evaluating A_3 and A_4 from (15) and substituting them into equations (12) we obtain the periodic orbit

$$\begin{aligned}
x(t) &= x_0\cos\big(\beta(t-t_0)\big)+(y_0/\alpha_3)\sin\big(\beta(t-t_0)\big),\\
y(t) &= y_0\cos\big(\beta(t-t_0)\big)-x_0\alpha_3\sin\big(\beta(t-t_0)\big).
\end{aligned}\tag{16}$$

This orbit is the ellipse

$$x^2+\frac{y^2}{\alpha_3^2}=x_0^2+\frac{y_0^2}{\alpha_3^2},\tag{17}$$

where $\alpha_3^2 = 1.69190..$ at the equilibrium points L_1 and L_4, and $\alpha_3^2 = 2.57633..$ at the equilibrium points L_2 and L_3.

6.3. FROM THE RESTRICTED TO THE FULL PROBLEM

We consider the planar 4-body problem in a rotating coordinates with one small particle, $m_4 = \varepsilon^2$, and the other three masses are equal, that without loss the generality we take $m_1 = m_2 = m_3 = 1$. We recall the following definition.

In a differential equations system, an equilibrium is called elementary (or non-degenerate) if its linearization matrix is non-singular. A periodic solution is called elementary (or non-degenerate) if no characteristic multipliers are unity; see Meiss (2007). In a Hamiltonian system, a periodic solution is elementary (or non-degenerate) if two and only tewo of the characteristic multipliers are unity. From Section 5 it is clear that all equilibria are elementary. Applying the results in Stoica (2016), we immediately obtain

Proposition 5. *All equilibria in the collinear restricted 4-body problem with logarithm potential persists as relative equilibria in the 4-body problem (with logarithm potential).*

Proposition 6. *Any elementary periodic solution of the collinear restricted* 4*-body problem with logarithm potential continue whose period is not a multiple of* 2π *persists as a periodic solution in the* 4*-body problem (with logarithm potential).*

Acknowledgements. The author was supported by a PhD fellowship at the University of Oradea, Romania.

REFERENCES

Abraham R. and Marsden J.: 1987, Foundations of Mechanics, Second Edition. Addison-Wesley Publishing Company.
Belmonte C., Boccaletti D. and Pucacco G.: 2007, *The Astrophysical Journal* **669**, 202.
Binney J. and Tremaine S.: 1987, Galactic Dynamics, Princeton NJ, Princeton University Press.
Castelli R. and S. Terracini S.: 2011, *Discrete and Continuous Dynamical Systems (DCDS)*, **31**, 1197.
Llibre J., Pașca D. and Valls C.: 2021, *Celestial Mechanics and Dynamical Astronomy*, **133**.
Meyer K. Hall G. and Offin D: 2009, Introduction to Hamiltonian Systems and the n-body Problem, Applied Mathematical Sciences, **90**, 2nd edition, Springer-Verlag, New York.
Meiss, J. D.: 2007, Differential Dynamical Systems, SIAM.
Miralda-Escude J. and Schwarzschild M.: 1989, *The Astrophysical Journal* **339**, 752.
Muscaș A-M., Pașca D. and Stoica C.: 2024, *New Astronomy*, **105**.
Muscaș A-M., Pașca D. and Stoica C.: 2024, *Celestial Mechanics and Dynamical Astronomy*, **136**.
Muscaș A-M., Pașca D. and Stoica C.: *Astrophysics and Space Science* **370**(3), 26, (2025)
Saha A. and Stoica C.: 2023, *Journal of Dynamics and Differential Equations*, 1–12.
Stoica C. and Font A.: 2003, *Journal of Physics A: Mathematical and General*, **36**, 7693.
Stoica C.: 2016, *Journal of Geometric Mechanics* **8**(3).
Szebehely V.: 1967, Theory of orbits, The restricted problem of three bodies, Academic Press, New York.
Valluri S. R., Wiegert P. A., Drozd J. and Silva M. D.: 2012, *Mon. Not. R. Astron. Soc.* **427**, 2392.
Valtonen M. and Karttunen H.: 2005, The three-body problem, Cambridge University Press.
Vidal C.: 2007, *Proyecciones*, **26**(2), 189.

Received on 2 April 2025