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EXPLORING BIANCHI TYPES COSMOLOGICAL MODELS IN THE FRAMEWORK OF MODIFIED $f(R)$ GRAVITY

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Abstract. In recent years, modified theories of gravity, particularly $f(R)$ gravity, have garnered much attention. This study investigates the exact vacuum field solutions for Bianchi type I, Bianchi type-III, and Kantowski-Sachs spacetimes within the $f(R)$ gravity metric theory. We assume a proportional relationship between the expansion scalar θ and shear scalar σ^2 and hence derive solutions of the form $X = Y^m$, where X and Y are the metric coefficients. The physical implications of these solutions are examined using relevant physical quantities. Further, we evaluate the functional Ricci scalar for each case. We observe that the average Hubble parameter $H(T)$ for the Bianchi type-I and type-III models decreases sharply with the progression of cosmic time, T, starting from a high initial value and then approaching a stable equilibrium. In the Kantowski-Sachs model, $H(T)$ shows a smoother and more uniform decline over time. This behavior aligns with the standard cosmological models, where the expansion rate of the universe slows down due to the gradual decrease in energy density. Moreover, the extremely high initial values of the expansion scalar, $\theta(T)$ as $T \to 0$ in all three models Bianchi type-I, type-III, and Kantowski-Sachs correspond well with the expansion phases of the early universe, such as inflation or the radiation-dominated era. The shear scalar $\sigma(T)^2$ consistently decreases with increasing cosmic time, indicating a trend toward isotropy as the universe expands. The volume scale factor $V(T)$ increases with time, capturing the continuous expansion which is a fundamental feature of the Big Bang cosmology. Additionally, the function $f(R)$ is well-behaved at low curvatures where $R \to 0$ but for large R in the limit of high curvature, $f(R) \to R$, in all of the three models.

Key words: Bianchi Type Metric, modified $f(R)$ gravity, Hubble parameter, expansion scalar, shear Scalar.

1. INTRODUCTION

The accelerated expansion as observed in the late-time universe has attracted significant attention in the recent years. Direct evidence for this cosmic acceleration is provided by high-redshift supernova experiments (Bennett *et al.*, 2003; Reddy *et al.*, 2009; Riess *et al.*, 2004), while indirect evidence comes from observations like cosmic microwave background fluctuations (Spergel *et al.*, 2003, 2007) and large-scale structure data (Tegmartk *et al.*, 2004). These findings have reshaped our understanding of the universe which is predominantly composed of dark matter and dark energy. Another compelling approach to explain this acceleration is provided through modifications of general relativity (GR).

Over the past decade, the $f(R)$ theory of gravity is treated as one of the most extensively studied modifications of General Relativity (GR). This theory provides a compelling gravitational framework as an alternative to dark energy. The suggestion is to achieve cosmic acceleration by replacing the EH action in General Relativity with $f(R)$, which is considered as a generalized function of the Ricci scalar. It has been demonstrated in Chiba *et al.* (2007) that modified $f(R)$ theory of gravity is mathematically equivalent to scalar-tensor theories of gravity, though it faces challenges regarding compatibility with solar system tests, when the scalar field propagates over these scales. The cosmic acceleration can be effectively explained by introducing the term $1/R$ which becomes significant at low curvature. Furthermore, studies in Capozziello *et al.* (2008), have shown that specific $f(R)$ models based on power-law solutions can fruitfully define both dust-dominated and dark energy phases of the universe. The $f(R)$ gravitational theory is often regarded as one of the most promising frameworks as it's cosmologically models are very significant. These models display curvature invariants of higher-order, expressed as functions of the Ricci scalar, and are effective in addressing the key cosmic phenomena. Several viable $f(R)$ models in Nojiri Odintsov (2010) have successfully explained the unification of early-time inflation with the late-time cosmic acceleration. Further, these models offer potential solutions to the dark matter problem. $f(R)$ gravity also provides other significant advantages, as discussed in Nojiri and Odintsov (2007). It provides a unified treatment of early inflation and late-time acceleration and also presents a framework for explaining the hierarchy problem in high-energy physics. Furthermore, it offers valuable insights into the transition phase of the universe from a decelerated to an accelerated phase. Hence, the $f(R)$ gravity theory has attracted much attention, and significant progress has been made in this area across various cosmological phenomena.

Within the framework of $f(R)$ gravity, wormhole geometries were constructed in Lobo and Oliveira (2009). The authors in Cognola *et al.* (2005) examined $f(R)$ gravity at the one-loop level in a de-Sitter universe and found that the one-loop effective action can be useful in understanding the nucleation rate of the constant curvature black holes. Additionally, spherically symmetric vacuum solutions within the framework of $f(R)$ are investigated in Multamäki and Vilja (2006). The authors in Multamäki and Vilja (2007) investigated perfect fluid solutions within the realm of $f(R)$ gravity. They demonstrated that the pressure and density do not uniquely determine the form of $f(R)$. In a recent work, Sharif and Shamir (2009) examined the non-vacuum static spherically symmetric solutions in this theory. Using the Noether symmetry approach, Capozziello *et al.* (2007) explored spherically symmetric solutions in $f(R)$ gravity theories. In Hollenstein and Lobo (2008), the authors analyzed exact solutions for static spherically symmetric spacetimes in $f(R)$ gravity coupled with non-linear electrodynamics. Studies on cylindrically symmetric vacuum solutions were made in Azadi *et al.* (2008), while Momeni and Gholizade (2009) provided an extension of these cylindrically symmetric solutions. Moreover, Reboucas and Santos (2009) studied Godel-type universes within the context of $f(R)$ gravity.

We have investigated static plane-symmetric vacuum solutions in $f(R)$ gravity (Sharif and Shamir, 2010a). The field equations are derived under the assumption that the scalar curvature is constant, which either vanishes or is non-zero. More recently, Babichev and Langlois (2009) also examined relativistic stars within the framework of this theory.

The Friedmann-Robertson-Walker (FRW) models are characterized by their spatial homogeneity and isotropy, hence are ideal for modeling the present universe in its large-scale nature. The early universe and its perfectly uniform character is widely debated. However, models involving anisotropic backgrounds, such as Bianchi type models, are more appropriate in defining the dynamics and characteristics of the early stages of the universe. Various authors have examined Bianchi type spacetimes in different contexts (Lorenz, 1982; Hanquin and Demaret, 1984; Tikekar and Patel, 1992; Yavuz and Yilmaz, 1996; Chakraborty *et al.*, 2003; Reddy *et al.*, 2009; Christodoulakis and Terzis, 2007; Bagora, 2009). For instance, Capozziello *et al.* (2009) analyzed the field equations for Bianchi type I spacetime within the framework of GR, by incorporating a perfect fluid. In Moussiaux *et al.* (1981), the authors derived exact solutions for a vacuum Bianchi type III model in presence of a cosmological constant, while Lorenz (1982) investigated the exact solutions for Bianchi type III spacetime effected by an electromagnetic field. Further, string cosmology in Bianchi type III spacetimes are explored in Xing-Xiang (2005), which focussed on the effects of bulk viscosity. The author assumed a proportional relationship between the expansion scalar and the shear scalar and hence derived the solutions. Wang (2005) examined string cosmological models with bulk viscosity within the Kantowski-Sachs spacetime. Upadhaya and Dave (2008) explored magnetized Bianchi type III massive string cosmological models within the framework of GR. More recently, Hellaby (2009) provided an overview of the furtherance in inhomogeneous models. It is found that the universe exhibits inhomogeneity on multiple scales.

Exploring Bianchi-type models within the context of alternative or modified theories of gravity presents an engrossing area of research. Kumar and Singh (2008) have examined the perfect fluid character in scalar-tensor model for Bianchi type-I spacetimes. Bianchi type-III cosmological models in scalar-tensor theory are analyzed in Singh and Agrawal (1991). In contrast, Adhav *et al.* (2009) derived an exact solution of the vacuum Brans-Dicke field equations for a spatially homogeneous and anisotropic model. Paul *et al.* (2009) recently investigated the FRW cosmologies within the framework of $f(R)$ gravity. We have studied the solutions for Bianchi types I and V spacetimes in $f(R)$ gravity (Sharif and Shamir, 2010,a).

In this paper, we intend to explore the vacuum field solutions for Bianchi type-I, type-III, and Kantowski-Sachs field spacetimes within the framework of the $f(R)$ gravitational theory. The field equations are deduced, where the expansion scalar θ is assumed to be proportional to the shear scalar σ. This results in the relationship $X = Y^m$. Here X and Y represent the metric coefficients. We use the geometric units $G = c = 1$. Here the cosmic time, T is expressed in geometric unit of length as mpc and $H(T)$ is expressed in mpc^{-1}. The expansion scalar $\theta(T)$ is expressed in mpc^{-1}. $f(R)$ and R have the same geometric units as mpc^{-2}. The shear scalar $\sigma(T)^2$ has the same geometric units as the Ricci scalar R as mpc^{-2}. The volume scale factor $V(T)$ is expressed in mpc^3. Here mpc is megapersec. We structure the paper as follows: Section 2 reveals the concise overview of the field equations in the background of the $f(R)$ gravity metric formalism. In Sections 3, 4, and 5, we derive the exact solutions to the field equations in Bianchi type-I, type-III, and Kantowski-Sachs field spacetimes. We also evaluate the relevant physical parameters and the functions of the Ricci scalar in the context of the solutions. Finally, we conclude with discussions of the results obtained.

2. MODIFIED $f(R)$- GRAVITY

In general relativity, the metric tensor is significant. A primary characteristic of General Relativity is that the Levi-Civita connection is determined by the metric tensor. This connection, however, ceases to exist if torsion is permitted in the theory, and the connection's reliance on the metric tensor disappears. This is the fundamental concept underlying several $f(R)$ theories of gravity.

When adopting the Levi-Civita connection, one derives metric $f(R)$ gravity. This method involves varying the action alone with respect to the metric tensor. The action for $f(R)$ gravity is presented by Sharif and Shamir (2009)

$$S = \int \sqrt{-g}\left(\frac{f(R)}{16\pi G} + \mathcal{L}_m\right) d^4x, \tag{1}$$

where $f(R)$ is a standard function of the Ricci scalar and $\mathcal{L}_m$ represents the matter Lagrangian. The field equations derived from this action are Sharif and Shamir (2009)

$$F(R)R_{\nu\mu} - \frac{1}{2}f(R)g_{\nu\mu} - \nabla_\nu\nabla_\mu F(R) + g_{\nu\mu}\,\Box\, F(R) = \kappa T_{\nu\mu}, \tag{2}$$

Here, $F(R) \equiv \frac{df(R)}{dR}\,\Box \equiv \nabla^\nu\nabla_\nu, \nabla_\nu$ means the covariant derivative and $T_{\nu\mu}$ denotes the standard minimally coupled stress-energy tensor derived from the Lagrangian $\mathcal{L}_m$. Now, contracting the field equations, which follows as Sharif and Shamir (2010a),

$$F(R)R - 2f(R) + 3\Box F(R) = \kappa T, \tag{3}$$

and in a vacuum, this reduces to become as,

$$F(R)R - 2f(R) + 3\Box F(R) = 0, \tag{4}$$

which implies this equation (4) as

$$f(R) = \frac{3\Box F(R) + F(R)R}{2}. \tag{5}$$

This establishes an important connection between $f(R)$ and $F(R)$, which will enable the field equations to be simplified and evaluate $f(R)$.

The general Bianchi type metric is considered as Sharif and Shamir (2009),

$$ds^2 = dt^2 - X^2(t)dr^2 - Y^2(t)\left(d\theta^2 + l^2(\theta)d\phi^2\right). \tag{6}$$

Here,

$$l^2(\theta) = \begin{cases} \theta^2, & \text{for } k = 0 \text{ (in Bianchi type-I metric)},\\ \sinh^2\theta, & \text{for } k = -1 \text{ (in Bianchi type-III metric)},\\ \sin^2\theta, & \text{for } k = 1 \text{ (in Kantowski-Sachs metric)}. \end{cases}$$

where k denotes the spatial curvature for the above three models being Euclidean, Semi-closed and Closed respectively.

3. BIANCHI TYPE-I METRIC AND FIELD EQUATIONS

We will derive exact solutions for Bianchi type-I spacetime within the framework of modified $f(R)$ theory of gravity. For brevity, we consider the vacuum field equations. The line element for Bianchi spacetime type I is expressed as Sharif and Shamir (2010):

$$ds^2 = dt^2 - X^2(t)dr^2 - Y^2(t)[d\theta^2 + \theta^2 d\phi^2], \tag{7}$$

Here, $X(t)$ and $Y(t)$ represent the cosmic scale factors. The Ricci scalar thereby takes the form as,

$$R = -2\left[\frac{\ddot{X}}{X} + \frac{2\ddot{Y}}{Y} + \frac{2\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2}\right], \tag{8}$$

where a dot represent the derivative with respect to the cosmic time t. We use equation (5), to derive the vacuum field equations asSharif and Shamir (2010),

$$\frac{F(R)R_{\nu\mu} - \nabla_\nu\nabla_\mu F(R)}{g_{\nu\mu}} = \frac{F(R)R - \Box F(R)}{4}. \tag{9}$$

One can view the equation (9) as a set of differential equations corresponding to the functions, X, Y and $F(t)$. It is found from equation (8), the combination as

$$X_\nu = \frac{F(R)R_{\nu\nu} - \nabla_\nu\nabla_\nu F(R)}{g_{\nu\nu}}, \tag{10}$$

which is independent of index ν. Thus, $X_\nu - X_\mu = 0$ for all ν and μ. As a result, $X_0 - X_1 = 0$ gives,

$$-\frac{2\ddot{Y}}{Y} + \frac{2\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{F}}{XF} - \frac{\ddot{F}}{F} = 0. \tag{11}$$

Also, $X_0 - X_2 = 0$ yield as,

$$-\frac{\ddot{X}}{X} - \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Y}\dot{F}}{YF} - \frac{\ddot{F}}{F} = 0. \tag{12}$$

Next, we will define some physical quantities which are helpful in solving these field equations.

The average scale factor a as well as the volume scale factor V is defined as

$$a = \sqrt[3]{XY^2},\ V = a^3 = XY^2. \tag{13}$$

Further, the average Hubble parameter, H and the deceleration parameters q are expressed as,

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y}\right). \tag{14a}$$

$$\begin{aligned} q &= -\frac{a\ddot{a}}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right) \\ &= -1 - \frac{3\left(\frac{\ddot{X}}{X} - \left(\frac{\dot{X}}{X}\right)^2 + 2\left(\frac{\ddot{Y}}{Y} - \left(\frac{\dot{Y}}{Y}\right)^2\right)\right)}{\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right)^2}. \end{aligned} \tag{14b}$$

The expansion scalar θ as well as the shear scalar σ is expressed as follows,

$$\theta = u^{\nu}_{;\nu} = \left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right), \tag{15}$$

$$\sigma^2 = \frac{1}{2}\sigma_{\nu\mu}\sigma^{\nu\mu} = \frac{1}{3}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)^2, \tag{16}$$

where

$$\sigma_{\nu\mu} = \frac{1}{2}\left(u_{\nu;\alpha}h^{\alpha}_{\mu} + u_{\mu;\alpha}h^{\alpha}_{\nu}\right) - \frac{1}{3}\theta h_{\nu\mu}, \tag{17}$$

$h_{\nu\mu} = g_{\nu\mu} - u_{\nu}u_{\mu}$ means the projection tensor and $u_{\nu} = \sqrt{g_{00}}\,(1,0,0,0)$ denotes the four-velocity in co-moving coordinates.

3.1. SOLUTIONS OF THE BIANCHI TYPE-I FIELD EQUATIONS

We obtain two differential equations denoted by equations (10) and (11) contain three unknowns: X, Y, and F. Therefore, an additional constraint is required to solve the equations. A physical condition is utilized where the expansion scalar θ is assumed to be proportional to the shear scalar σ. This leads to,

$$X = Y^m. \tag{18}$$

By using this condition, equation (11) and equation (12) take the form as

$$-\frac{2\ddot{Y}}{Y} + 2m\frac{\dot{Y}^2}{Y^2} + m\frac{\dot{Y}\dot{F}}{YF} - \frac{\ddot{F}}{F} = 0, \tag{19}$$

$$(m+1)\frac{\ddot{Y}}{Y} + \left(m^2 - 2m - 1\right)\frac{\dot{Y}^2}{Y^2} - \frac{\dot{Y}\dot{F}}{YF} + \frac{\ddot{F}}{F} = 0. \tag{20}$$

By adding these two equations, we obtain

$$\frac{\ddot{Y}}{Y} + (m+1)\frac{\dot{Y}^2}{Y^2} + \frac{\dot{Y}\dot{F}}{YF} = 0. \tag{21}$$

By using the power law relation for F and a to solve this equation, we have (Sharif and Shamir, 2009),

$$F = \beta a^k, \tag{22}$$

where β is the constant of proportionality, k is any integer and a is can be written as

$$a = Y^{\frac{m+2}{3}} \tag{23}$$

Thus, for $k = 3$, we get

$$F = \beta\, Y^{m+2}. \tag{24}$$

By using this relation into equation (21), which follows as

$$\frac{\ddot{Y}}{Y}+(2m+3)\frac{\dot{Y}^2}{Y^2}=0. \tag{25}$$

After putting $\dot{Y}=g(Y)$ in this equation, we obtain

$$\frac{dg^2}{dY}+\left(\frac{4m+6}{Y}\right)g^2=0, \tag{26}$$

which gives the solution as

$$g^2=\frac{c_1}{Y^{4m+6}}, \tag{27}$$

where c_1 is the constant of integration. Hence, the solution becomes

$$ds^2=\left(\frac{dt}{dY}\right)^2 dY^2-Y^{2n}dr^2-Y^2\left(d\theta^2+\theta^2 d\phi^2\right), \tag{28}$$

which can be written in the form as

$$ds^2=\frac{1}{c_1}Y^{4m+6}dY^2-Y^{2m}dr^2-Y^2\left(d\theta^2+\theta^2 d\phi^2\right), \tag{29}$$

and using the transformations, $Y=T, \theta=\Theta$, $r=R$, and $\phi=\Phi$, we find,

$$ds^2=\frac{1}{c_1}T^{4m+6}dT^2-T^{2m}dR^2-T^2\left(d\Theta^2+\Theta^2 d\Phi^2\right). \tag{30}$$

The average Hubble parameter, H, and the deceleration parameter, q, are obtained as

$$H=\frac{\sqrt{c_1}\,(m+2)}{3T^{2m+4}}, \tag{31a}$$

$$q=-1+\frac{3(2m+4)}{\sqrt{c_1}(m+2)}T^{2m+3}, \tag{31b}$$

while the volume scale factor V becomes as

$$V=T^{m+2}. \tag{32}$$

The expansion scalar θ can be written as

$$\theta=\frac{\sqrt{c_1}\,(m+2)}{T^{2m+4}}, \tag{33}$$

while the shear scalar σ becomes as

$$\sigma^2=\frac{c_1\,(m-1)^2}{3T^{4m+8}}. \tag{34}$$

The Ricci scalar function, $f(R)$ is obtained by using equation (5), hence we have,

$$f(R)=\frac{\beta T^{m+2}R}{2}. \tag{35}$$

From equation (8), we observe that,

$$R = 2c_1(m^2+6m+5)T^{-4m-8}. \tag{36}$$

Hence, $f(R)$ can be expressed as a function of R only,

$$f(R) = \frac{\beta}{2}\left[2c_1\left(m^2+6m+5\right)\right]^{\frac{m+2}{4m+8}} R^{\frac{3}{4}}. \tag{37}$$

Figures 1(a), 1(b) and 2 represent the average Hubble parameter of equations 31(a), 51(a), and 66(a) respectively. Figures 3(a), 3(b), and 4 represent the deceleration parameter of equations 31(b), 51(b), and 66(b) respectively. Figures 5(a), 5(b), and 6 represent the expansion scalar θ of equations 33, 52 and 67, respectively. Figures 7(a), 7(b), and 8 represent the shear scalar σ^2 of equations 34, 53, and 68, respectively. Figures 9(a), 9(b), and 10 represent the volume scale factor V of equation 32 for all three cases, respectively. Figures 11(a), 11(b), and 12 represent the function of Ricci scalar $f(R)$ of equations 37, 54, and 69, respectively, for all the three cases of physical parameters of Bianchi type-I, Bianchi type-III and Kantowski-Sachs model.

4. BIANCHI TYPE-III WITH THEIR FIELD EQUATIONS

We will derive the solution of the Bianchi spacetime type III, in modified $f(R)$ theory of gravity for vacuum field equations. The line element of Bianchi spacetime type III is expressed as Upadhaya and Dave (2008),

$$ds^2 = dt^2 - X^2(t)dr^2 - Y^2(t)\left[d\theta^2 + \sinh^2\theta d\phi^2\right], \tag{38}$$

where, $X(t)$ and $Y(t)$ are cosmic scale factors. The Ricci scalar for this spacetime can be written as

$$R = -2\left[\frac{\ddot{X}}{X} + \frac{2\ddot{Y}}{Y} + \frac{2\dot{X}\dot{Y}}{XY} - \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2}\right], \tag{39}$$

where, dot means the derivative with respect to cosmic time T. By using equation (10), the vacuum field equations can be written as,

$$-\frac{2\ddot{Y}}{Y} + \frac{2\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{F}}{XF} - \frac{\ddot{F}}{F} = 0, \tag{40}$$

$$-\frac{\ddot{X}}{X} - \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} - \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Y}\dot{F}}{YF} - \frac{\ddot{F}}{F} = 0. \tag{41}$$

4.1. SOLUTIONS OF THE BIANCHI TYPE-III FIELD EQUATIONS

An additional constraint is required to resolve these equations. By assuming the same physical constraint, that θ, the expansion scalar being proportional to σ, the shear scalar, we derive

$$-\frac{2\ddot{Y}}{Y}+2m\frac{\dot{Y}^2}{Y^2}+m\frac{\dot{Y}\dot{F}}{YF}-\frac{\ddot{F}}{F}=0, \tag{42}$$

$$(m+1)\frac{\ddot{Y}}{Y}+(m^2-2m-1)\frac{\dot{Y}^2}{Y^2}-\frac{\dot{Y}\dot{F}}{YF}+\frac{\ddot{F}}{F}=-\frac{1}{Y^2}. \tag{43}$$

By adding these above equation (42) and equation (43), we get

$$\frac{\ddot{Y}}{Y}+\left(m+1\right)\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Y}\dot{F}}{YF}=-\frac{1}{(m-1)Y^2}. \tag{44}$$

We use the power law relationship between Y and F and hence find,

$$F=\beta Y^{m+2}. \tag{45}$$

Further, the differential equation including an unknown parameter is obtained as,

$$\frac{\ddot{Y}}{Y}+(2m+3)\frac{\dot{Y}^2}{Y^2}=-\frac{1}{(m-1)Y^2}. \tag{46}$$

After putting $\dot{Y}=g(Y)$ in this equation, we obtain,

$$\frac{dg^2}{dY}+\left(\frac{4m+6}{Y}\right)g^2=-\frac{2}{(m-1)Y}, \tag{47}$$

which leads to the solution of above equation (47), we get

$$g^2=\frac{c_2}{Y^{4m+6}}-\frac{1}{(m-1)(2m+3)}, \tag{48}$$

where, c_2 is the constant of integration and hence, the solution becomes as

$$ds^2=\left(\frac{1}{\frac{c_2}{Y^{4m+6}}-\frac{1}{(m-1)(2m+3)}}\right)dY^2-Y^{2m}dr^2-Y^2(d\theta^2+\sinh^2\theta d\phi^2), \tag{49}$$

which turn out be in the form of

$$ds^2=\left(\frac{1}{\frac{c_2}{T^{4m+6}}-\frac{1}{(m-1)(2m+3)}}\right)dT^2-T^{2m}dR^2-T^2(d\Theta^2+\sinh^2\theta d\phi^2), \tag{50}$$

where $Y=T, r=R, \theta=\Theta$, and $\phi=\Phi$.

Here, the average Hubble parameter, H, and the deceleration parameter, q, can be obtained as

$$H=\left(\frac{m+2}{3}\right)\left[\frac{c_2}{T^{4m+8}}-\frac{1}{(m-1)(2m+3)T^2}\right]^{1/2}. \tag{51a}$$

$$q=-1+\frac{1}{2\cdot\left(\frac{m+2}{3}\right)\left(\frac{c_2}{T^{4m+8}}-\frac{1}{(m-1)(2m+3)T^2}\right)^{3/2}} \times\left[c_2(4m+8)T^{-(4m+9)}-\frac{2}{(m-1)(2m+3)T^3}\right]. \tag{51b}$$

The volume scale factor V is identical to that of the Bianchi spacetime type I. Further θ, the expansion scalar is defined as,

$$\theta=(m+2)\left[\frac{c_2}{T^{4m+8}}-\frac{1}{(m-1)(2m+3)T^2}\right]^{1/2}. \tag{52}$$

Now, σ, the shear scalar can be obtained as

$$\sigma^2=\frac{1}{3}(m-1)^2\left[\frac{c_2}{T^{4m+8}}-\frac{1}{(m-1)(2m+3)T^2}\right]^{1/2}. \tag{53}$$

The Ricci scalar function, $f(R)$ is obtained by using equation (5),

$$f(R)=\frac{\beta}{2}\left[T^{m+2}R-\frac{3(2m^2+7m+6)}{2m^2+m-3}T^m\right]. \tag{54}$$

It follows from equation (39) that

$$R=2\left[\frac{c_2(m^2+6m+5)}{T^{4m+8}}+\frac{1}{T^2}\left(\frac{3m^2+2m-2}{2m^2+m-3}\right)\right], \tag{55}$$

which makes it clear that $f(R)$ can't be written directly in terms of R. When $m=-5$, on the other hand, $f(R)$ reduces as,

$$f(R)=\frac{\beta}{12\sqrt{3}}R^{5/2},$$

which gives a function of R only for $f(R)$.

5. KANTOWSKI-SACHS METRIC & FIELD EQUATIONS

The line element of Kantowski-Sachs spacetime can be expressed as Wang (2005),

$$ds^2=dt^2-X^2(t)dr^2-Y^2(t)\left[d\theta^2-\sin^2\theta d\phi^2\right], \tag{56}$$

where X, Y represent the cosmic scale factors. R, the corresponding Ricci scalar, is obtained as,

$$R=-2\left[\frac{\ddot{X}}{X}+\frac{2\ddot{Y}}{Y}+\frac{2\dot{X}\dot{Y}}{XY}+\frac{1}{Y^2}+\frac{\dot{Y}^2}{Y^2}\right], \tag{57}$$

Here, a dot represents the derivative with respect to t, the cosmic time. The field equations in vacuum corresponding to Kantowski-Sachs spacetime can be written as,

$$-\frac{2\ddot{Y}}{Y}+\frac{2\dot{X}\dot{Y}}{XY}+\frac{\dot{X}\dot{F}}{XY}-\frac{\ddot{F}}{F}=0, \tag{58}$$

$$-\frac{\ddot{X}}{X}-\frac{\ddot{Y}}{Y}+\frac{\dot{X}\dot{Y}}{XY}+\frac{1}{Y^2}+\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Y}\dot{F}}{YF}-\frac{\ddot{F}}{F}=0. \tag{59}$$

5.1. SOLUTIONS OF THE KANTOWSKI-SACHS FIELD EQUATIONS

We have use equation (18) so that the field equations can be obtain as

$$-\frac{2\ddot{Y}}{Y}+2m\frac{\dot{Y}^2}{Y^2}+m\frac{\dot{Y}\dot{F}}{YF}-\frac{\ddot{F}}{F}=0, \tag{60}$$

$$(m+1)\frac{\ddot{Y}}{Y}+(m^2-2m-1)\frac{\dot{Y}^2}{Y^2}-\frac{\dot{Y}\dot{F}}{YF}+\frac{\ddot{F}}{F}=\frac{1}{Y^2}. \tag{61}$$

After adding these equation (60) and equation (61), we have

$$\frac{\ddot{Y}}{Y}+(m+1)\frac{\dot{Y}^2}{Y^2}+\frac{\dot{Y}\dot{F}}{YF}=\frac{1}{(m-1)Y^2}. \tag{62}$$

By using equation (24), we have obtained one differential equation with one unknown value,

$$\frac{\ddot{Y}}{Y}+(2m+3)\frac{\dot{Y}^2}{Y^2}=\frac{1}{(m-1)Y^2}, \tag{63}$$

which implies

$$\dot{Y}^2=\frac{c_3}{Y^{4m+6}}+\frac{1}{(m-1)(2m+3)}, \tag{64}$$

where c_3 is the constant of integration and hence, the solution can be obtain as

$$ds^2=\left(\frac{1}{\frac{c_3}{T^{4m+6}}}+\frac{1}{\frac{1}{(m-1)(2m+3)}}\right)dT^2$$

$$-T^{2m}dR^2-T^2(d\theta^2+\sin^2\theta d\phi^2), \tag{65}$$

where $Y=T, r=R, \theta=\Theta$, and $\phi=\Phi$.

Now, the average Hubble parameter, H and the deceleration parameter, q, can be written as

$$H=\left(\frac{m+2}{3}\right)\left[\frac{c_3}{T^{4m+8}}+\frac{1}{(m-1)(2m+3)T^2}\right]^{1/2}. \tag{66a}$$

$$q = -1 + \frac{1}{2\cdot\frac{m+2}{3}\cdot\left(\frac{c_3}{T^{4m+8}} + \frac{1}{(m-1)(2m+3)T^2}\right)^{3/2}} \times \left[c_3(4m+8)T^{-(4m+9)} + \frac{2}{(m-1)(2m+3)T^3}\right]. \tag{66b}$$

Although, the volume scale factor is the same as it is for Bianchi types I and III spacetimes. For the expansion scalar θ can be obtain as

$$\theta = (m+2)\left[\frac{c_3}{T^{4m+8}} + \frac{1}{(m-1)(2m+3)T^2}\right]^{1/2}. \tag{67}$$

However, the shear scalar σ is given by

$$\sigma^2 = \frac{1}{3}(m-1)^2\left[\frac{c_3}{T^{4m+8}} + \frac{1}{(m-1)(2m+3)T^2}\right]. \tag{68}$$

The function of Ricci scalar $f(R)$ can be written as

$$f(R) = \frac{\beta}{2}\left[T^{m+2}R + \frac{3(2m^2+7m+6)}{2m^2+m-3}T^m\right], \tag{69}$$

While, the Ricci scalar R turn out to be

$$R = 2\left[\frac{c_3(m^2+6m+5)}{T^{4m+8}} - \frac{1}{T^2}\left(\frac{(3m^2+2m-2)}{2m^2+m-3}\right)\right]. \tag{70}$$

When $m=-5$ then $f(R)$ can be written as:

$$f(R) = \frac{\beta}{\sqrt{48}}|-R|^{2+1/2}.$$

6. CONCLUDING REMARKS

Interpretation from figures: Figures 1(a) and 1(b) indicate that the average Hubble parameter H, for the Bianchi type-I and Bianchi type-III models, respectively, decreases steeply as cosmic time T increases, starting from a very high initial value, to a stable equilibrium. In Figure 2, for the Kantowski-Sachs model, we observe that as cosmic time progresses, $H(T)$ becomes more uniform and decreases steadily. This model is consistent with the standard cosmological model, where the expansion rate of the universe decreases over time because of the diminution of energy density. Figures 2(a) and 2(b) indicate that the deceleration parameter q, for the Bianchi type-I and Bianchi type-III models, respectively, increases steadily as cosmic time T increases, starting from a very low initial value, to a stable equilibrium. In Figure 2, for the Kantowski-Sachs model, we observe that as cosmic time progresses, q

becomes more uniform. The extremely high value of the expansion scalar $\theta(T)$ as $T \to 0$, in Figures 5(a), 5(b), and 6 (Bianchi type-I, type-III and Kantowski-Sachs models respectively) are in parity with the rapid expansion phase of the universe, such as inflation or the radiation-dominated era. During this time, the volume of the universe increases exponentially. The rapid drop in $\theta(T)$ when $T > 0.1$, suggests a deceleration of the cosmic expansion, due to the transition to the matter-dominated era, where gravity slows the expansion rate. For $T > 1$, $\theta(T)$ becomes stable at small values which suggests that the universe transited to a phase of slower expansion, corresponding to a dark energy-dominated era. Figures 7-8 reveal that the shear scalar σ^2 decreases with increasing cosmic time T, thereby indicating that the universe becomes more isotropic as it expands. From Figures 9-10, we find that the volume scale factor V increases with cosmic time T, reflecting the ongoing expansion of the universe, which being a fundamental feature of the Big Bang cosmology. Figures 11-12 indicate that $f(R)$ is well-behaved at low curvatures where $R \to 0$ but for large R in the limit of high curvature, $f(R) \to R$, in all of three models.

The main objective of this paper is to investigate certain Bianchi-type cosmological models within the framework of metric for $f(R)$ gravity theory. The exact solutions of the field equations in vacuum are derived for Bianchi spacetimes type I, III, and for Kantowski-Sachs spacetimes. Initially the field equations appear to be complex, but they simplify under certain specific assumptions. The primary key assumption is that θ, the expansion scalar is proportional to σ, the shear scalar. This give rise to the relation $X = Y^m$, where X, Y, represent the metric coefficients, and m, the arbitrary constant. Next, a power-law relationship between a and F is used to deduce the solution. Key physical quantities associated with the cosmological solutions, such as θ, the expansion scalar, σ^2, the shear scalar and H, the average Hubble parameter, are evaluated. Although this paper does not analyze these parameters in the light of solar system constraints, it would be worthwhile to assess their consistency with Wilkinson Microwave Anisotropy Probe (WMAP) data. Further, these solutions can be compared with the Lemaître-Tolman cosmology (Hellaby, 2009), which describes inhomogeneities in the universe on many scales.

In each case, the general form $f(R)$, the Ricci scalar function is derived. For certain specific cases, it isfurther observed that the $f(R)$ models are expressed as the root powers of R, the Ricci scalar. It has been demonstrated in Capozziello *et al.* (2009) that the model $f(R) = R^{3/2}$ fits well with cosmological observations. Further, flat rotation curves for galaxies can be obtained, which are consistent with solar system tests. However, further research is necessary to test the viability of the models as well as being consistent with the constraints in solar system.

The cosmological models for the universe as described in equations (30), (50), and (65) are found to be non-singular when $T = 0$. At this point, these physical parameters such as H, the Hubble parameter, θ, the expansion scalar and σ, the shear scalar, all diverge, while V, the volume scale factor becomes zero. The generalalized form of R, the Ricci scalar remains finite, but the spatial components of the metric tend to vanish when $T = 0$. When $m = -2$, this expansion comes to a halt in all models. These models indicate that over time, the expansion of the universe slows down and ultimately stops, leading to isotropy. This is consistent with the isotropic condition, $\frac{\sigma^2}{\theta} \to 0$, when $T \to \infty$, which is also satisfied in all the cases. Therefore, it is concluded that the models begin with their expansion from the volume state of zero, at $T = 0$, and this volume increases progressively with time.

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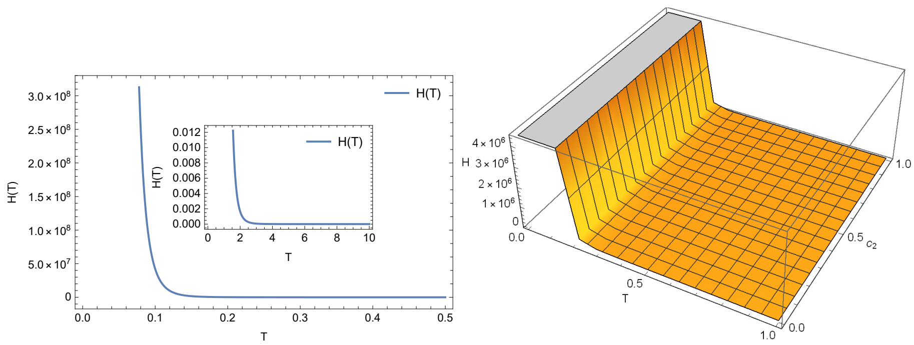


Fig. 1 – a (first row) The average Hubble parameter $H(mpc^{-1})$ versus cosmic time $T(mpc)$ for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b (second row) The average Hubble parameter $H(mpc^{-1})$ versus cosmic time $T(mpc)$ and c_2 for the Bianchi type-III in 3D, with the constant value of $m = 2$.

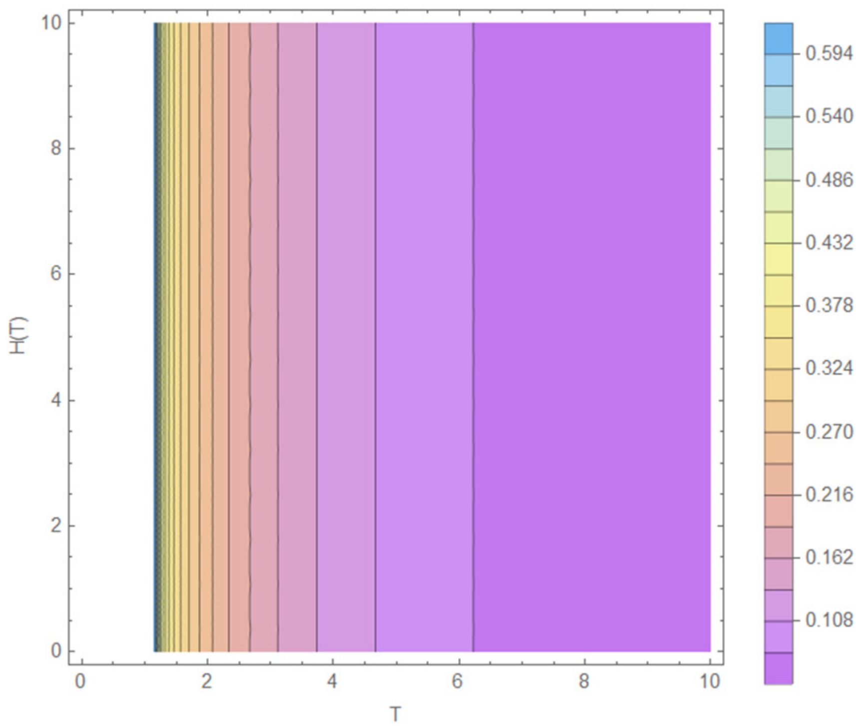


Fig. 2 – Density plot for the average Hubble parameter $H(mpc^{-1})$ versus cosmic time $T(mpc)$, for the Kantowski-Sachs with the constant values of $c_3 = 1, \ m = 2$.

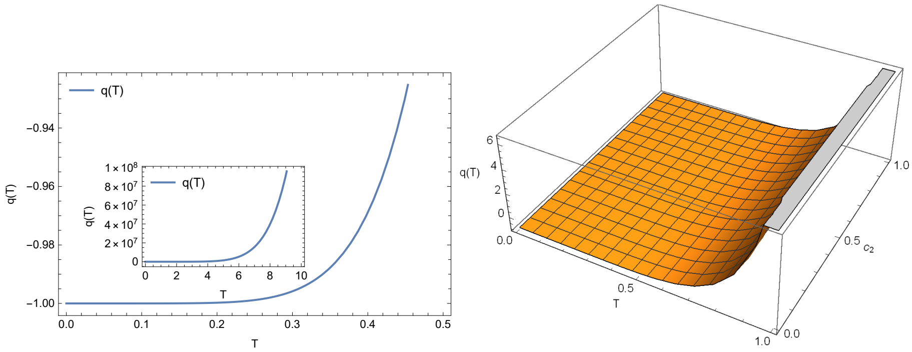


Fig. 3 – a (left) The deceleration parameter q versus cosmic time T for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b (right) The deceleration parameter q versus cosmic time T and c_2 for the Bianchi type-III in 3D, with the constant value of $m = 2$.

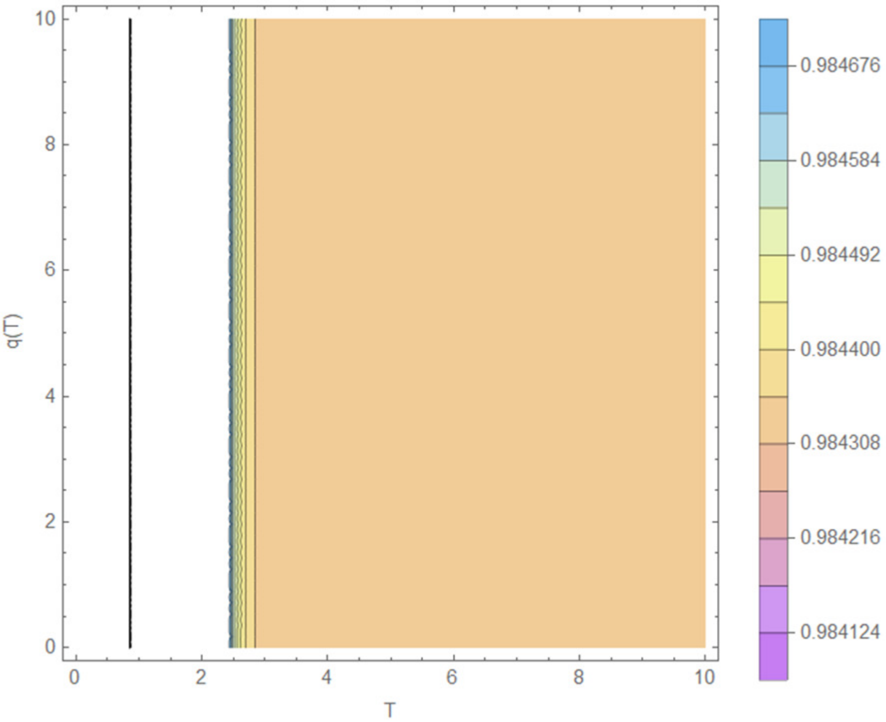


Fig. 4 – Density plot for the deceleration parameter q versus cosmic time T, for the Kantowski-Sachs with the constant values of $c_3 = 1, \ m = 2$.

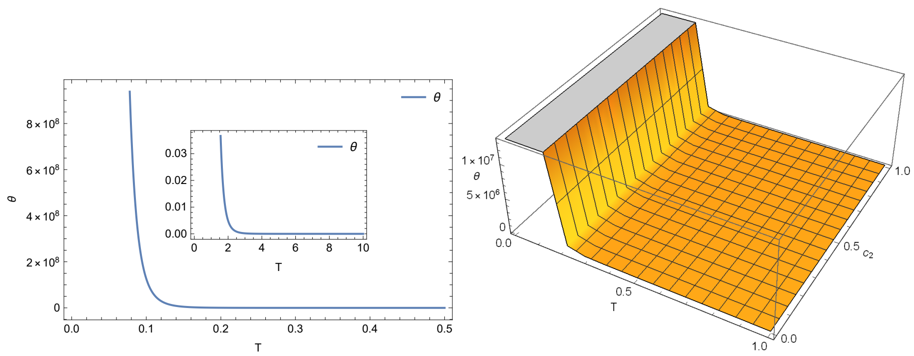


Fig. 5 – a (left) The expansion scalar $\theta(mpc^{-1})$ versus cosmic time $T(mpc)$ for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b (right) The expansion scalar $\theta(mpc^{-1})$ versus cosmic time $T(mpc)$ and c_2 Bianchi type-III in 3D with the constant value of $m = 2$

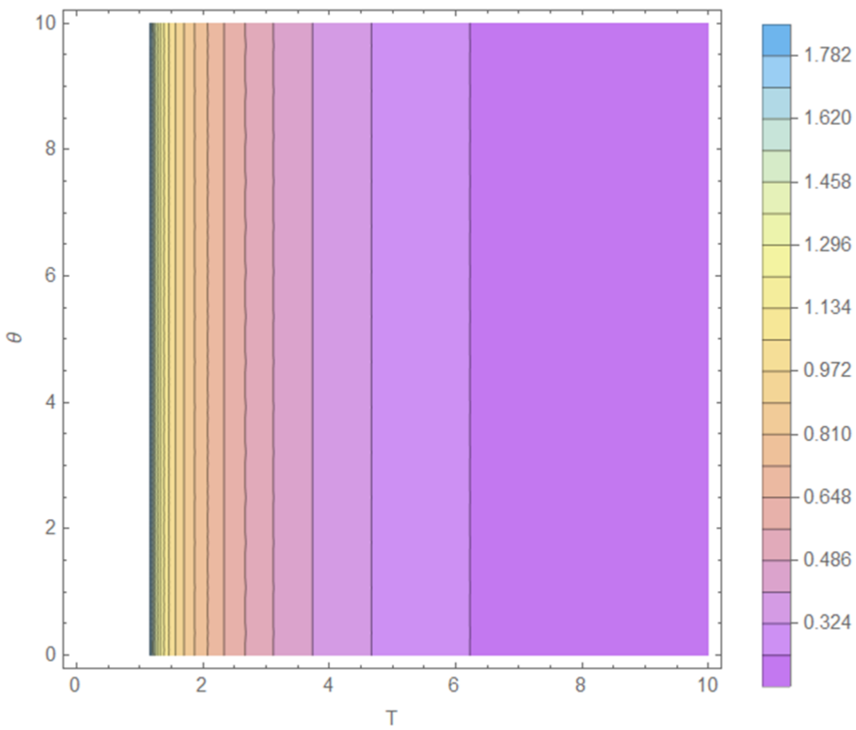


Fig. 6 – Density plot for the expansion scalar $\theta(mpc^{-1})$ versus cosmic time $T(mpc)$, for the Kantowski-Sachs with the constant values of $c_3 = 1, \ m = 2$.

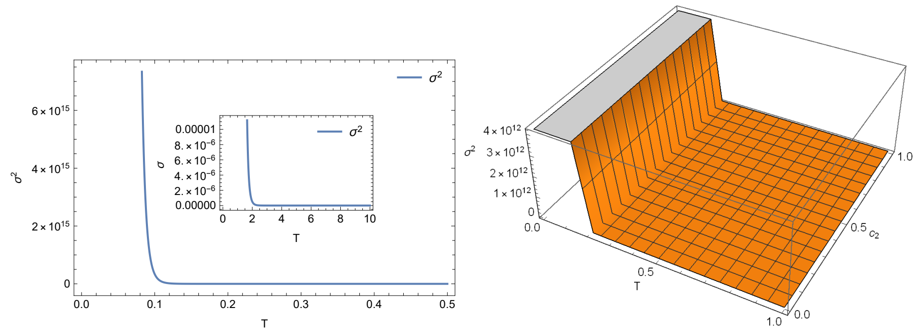


Fig. 7 – (left) The shear scalar $\sigma^2(mpc^{-2})$ versus cosmic time T for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b (right) The shear scalar $\sigma^2(mpc^{-2})$ versus cosmic time T and arbitrary constant c_2 Bianchi type-III in 3D with the constant value of $m = 2$.

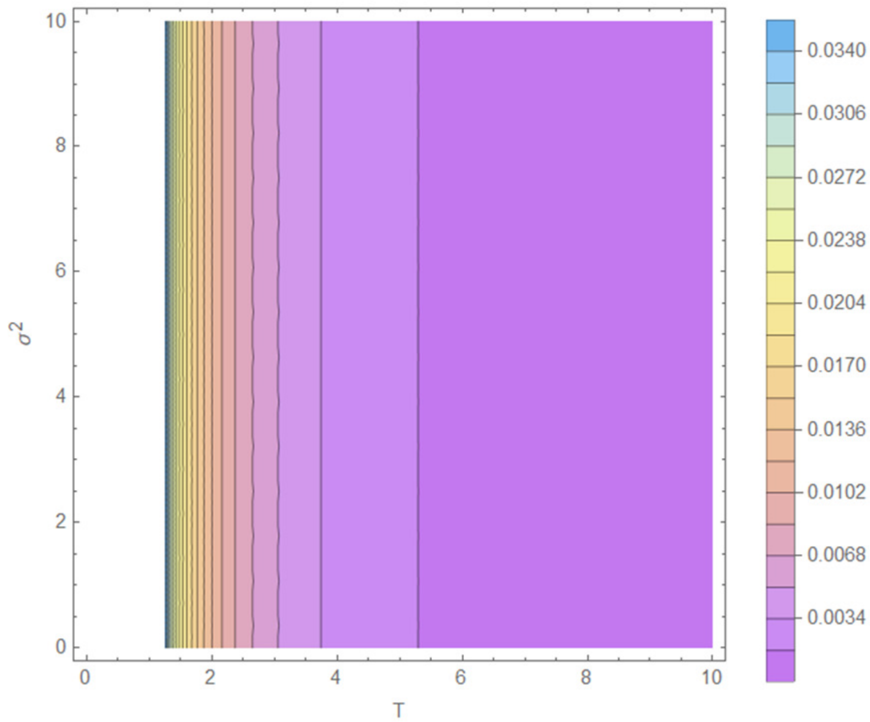


Fig. 8 – Density plot for the shear scalar $\sigma^2(mpc^{-2})$ versus cosmic time $T(mpc)$, for the Kantowski-Sachs with the constant values of $c_3 = 1, \; m = 2$.

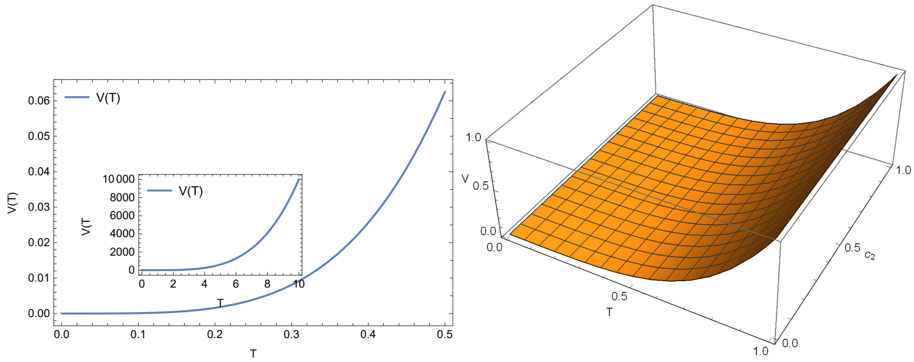


Fig. 9 – a (left) The volume scale factor $V(mpc^3)$ versus cosmic time $T(mpc)$ for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b. (right) The volume scale factor $V(mpc^3)$ versus cosmic time $T(mpc)$ and arbitrary constant c_2 Bianchi type-III in 3D with the constant values of $m = 2$.

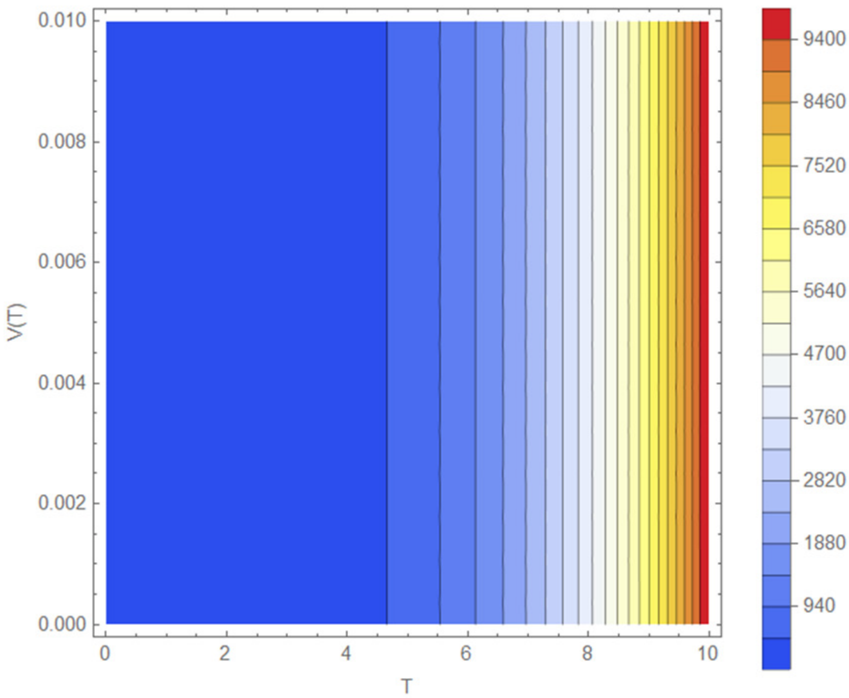
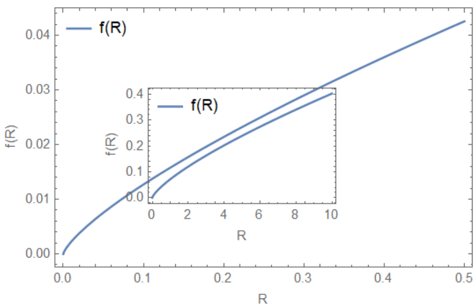


Fig. 10 – Density plot for the volume scale factor $V(mpc^3)$ versus cosmic time $T(mpc)$, for the Kantowski-Sachs with the constant values of $c_3 = 1, \; m = 2$.



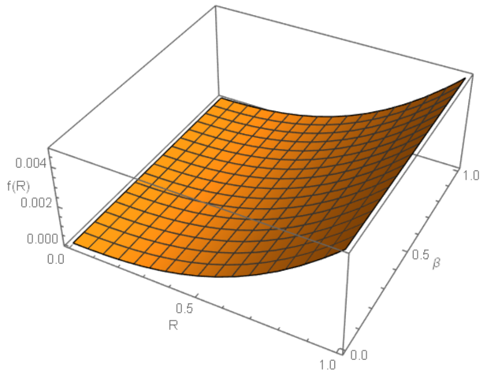


Fig. 11 – a (left) The function of Ricci scalar $f(R)(mpc^{-2})$ against Ricci scalar $R(mpc^{-2})$ for the Bianchi type-I in 2D with the constant values of $c_1 = 0.1, \quad m = 2$.
b (right) The function of Ricci scalar $f(R)(mpc^{-2})$ against Ricci scalar $R(mpc^{-2})$ and β for Bianchi type-III in 3D with the constant values of $c_2 = 0.5, \ m = 2$.

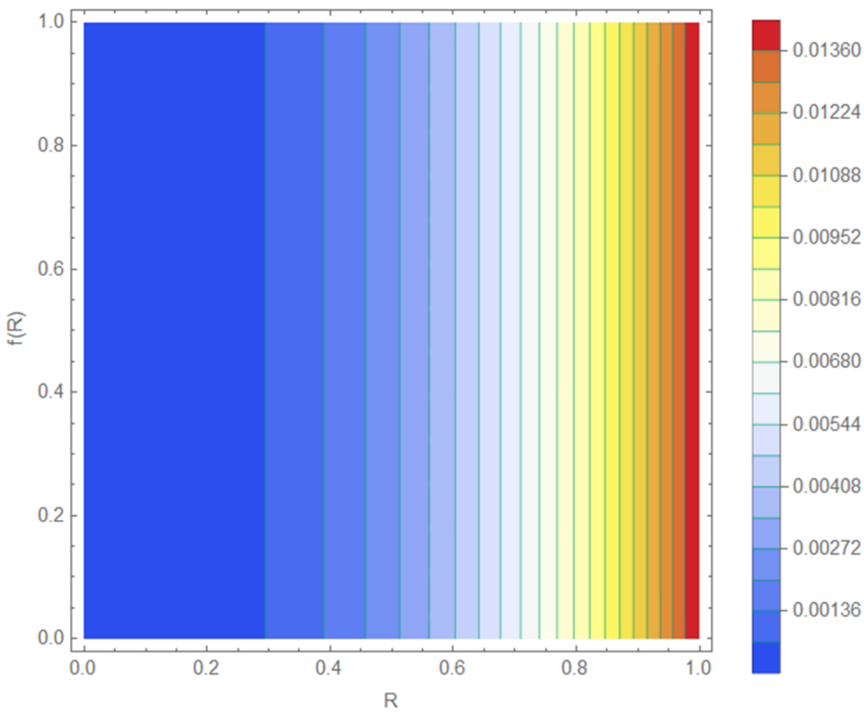


Fig. 12 – Density plot for the function of Ricci scalar $f(R)(mpc^{-2})$ against Ricci scalar $R(mpc^{-2})$, for the Kantowski-Sachs with the constant values of $\beta = 0.1, \quad m = 2$.