STABILITY OF TRIANGULAR EQUILIBRIUM POINTS FOR THE ELLIPTIC RESTRICTED THREE-BODY PROBLEM WITH DRAG

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Abstract. In this paper we generalize Rein's model for the elliptic restricted three-body problem (ER3BP) by taking into account drag. We determine the equations of motion, the stationary points and we study the linear stability of motion in the sense of Lyapunov around L_4 as a function of the mass parameter and the eccentricity of primaries. Applications to the Earth-Moon system are also presented, with trajectories computed around the L_4 equilibrium point.

Key words: elliptical restricted three-body problem; Rein's averaged method; drag force.

1. INTRODUCTION

The planar ER3BP consists of studying the motion of an infinitesimal mass point P under the gravitational attraction of two bodies, P_1 and P_2. Let m_1 and m_2 be the masses of P_1 and P_2, respectively.

P_1 and P_2 describe elliptic orbits with a common focus in the center of mass O, eccentricity e, semi-major axes a_1 and a_2, and mean motion n. The motion of the point P takes place in the orbital plane of P_1 and P_2.

Let us consider a uniformly rotating frame $O\xi\eta$ that rotates around O with the constant angular velocity n and let the coordinates of the three points be $P_1(\xi_1,\eta_1)$, $P_2(\xi_2,\eta_2)$ and $P(\xi,\eta)$.

In this rotating frame, P_1 and P_2 describe closed curves around the "mean" points $\overline{P}_1(-a_1,0)$ and $\overline{P}_2(a_2,0)$, respectively. The positions of P_i, $i=1,2$, are given by Rein (1940):

$$\begin{aligned}\xi_i &= (-1)^i\, a_i(1-e^2)\tfrac{\cos(v-nt)}{1+e\cos v},\\ \eta_i &= (-1)^i\, a_i(1-e^2)\tfrac{\sin(v-nt)}{1+e\cos v},\end{aligned} \tag{1}$$

where v stands for the true anomaly, and

$$a_1 = \frac{am_2}{m_1+m_2}, \qquad a_2 = \frac{am_1}{m_1+m_2}, \tag{2}$$

a being the distance between $\overline{P}_1$ and $\overline{P}_2$.

The planar motion of the infinitesimal mass point P with respect to $O\xi\eta$ is described by the differential equations:

$$\ddot{\xi} - 2n\dot{\eta} = U_\xi \qquad \ddot{\eta} + 2n\dot{\xi} = U_\eta, \tag{3}$$

where subscripts mark the corresponding partial derivatives, and:

$$U = \frac{n^2}{2}(\xi^2+\eta^2) + \frac{k^2m_1}{r_1} + \frac{k^2m_2}{r_2}. \tag{4}$$

In the above formulas, k is Gauss' constant and r_1, r_2, are given by:

$$r_i^2 = (\xi-\xi_i)^2 + (\eta-\eta_i)^2, \quad i = 1,2. \tag{5}$$

2. REIN'S SCHEME

The force function given by (4) and used in (3) is an explicit function of t, as the coordinates (ξ_i, η_i), $i = 1,2$, are given by (1). Therefore, the differential system of equations (3) does not have a first integral analogous to Jacobi's integral for the circular restricted three-body problem. For this reason, several authors have proposed simplifying schemes for the ER3BP. In this paper we'll use one of these methods, called Rein's "semi-averaging" method (Rein (1940)).

The coordinates (ξ_i, η_i), $i = 1,2$ in (1) can be expressed by means of infinite power series of the eccentricity e, as follows:

$$\xi_i = (-1)^i a_i\{1 + e[-\tfrac{1}{2}e - (1+\tfrac{3}{8}e^2)\cos(nt) + \tfrac{1}{2}e\cos(2nt) + \tfrac{3}{8}e^2\cos(3nt) + \ldots]\},$$

$$\eta_i = (-1)^i a_i e[(2-\tfrac{3}{8}e^2)\sin(nt) + \tfrac{1}{4}e\sin(2nt) + \tfrac{7}{24}e^2\sin(3nt) + \ldots], i = 1,2. \tag{6}$$

For the points P_1 and P_2, if e is small enough to neglect e^j for $j \geq 2$, we have finite expressions of their coordinates and these are explicit functions of t. Let us denote these coordinates by $\overline{\xi}_i, \overline{\eta}_i$. We have:

$$\overline{\xi}_i = (-1)^i a_i(1 - e\cos(nt)) \qquad \overline{\eta}_i = 2(-1)^i a_i e\sin(nt) \quad i = 1,2. \tag{7}$$

The above equations represent ellipses having the centers at $\overline{P}_1$ and $\overline{P}_2$, respectively, and the semi-major axes parallel to the $O\xi\eta$-axes. The lengths of the semi-major axes A_i and B_i, $i=1,2$ of these ellipses are given by:

$$\begin{aligned} A_i &= 2a_i e \\ B_i &= a_i e, \quad i=1,2. \end{aligned} \tag{8}$$

In Rein's scheme, we replace the force function U by its time-averaged function $\overline{U}$:

$$\overline{U} = \frac{n^2}{2}(\overline{\xi}^2+\overline{\eta}^2) + \frac{1}{T}\int_{t_0}^{t_0+T}(\frac{k^2 m_1}{r_1}+\frac{k^2 m_2}{r_2})\mathrm{d}t, \tag{9}$$

with $T=2\pi/n$, r_1 and r_2 given by (5), and ξ_i,η_i replaced by $\overline{\xi}_i,\overline{\eta}_i$, respectively.

In order to emphasize that the motion of P is governed by the force function $\overline{U}$, we rename the frame $O\xi\eta$ as $O\overline{\xi}\overline{\eta}$. The averaged function $\overline{U}$ can be rewritten as:

$$\overline{U} = \frac{n^2}{2}(\overline{\xi}^2+\overline{\eta}^2) + \frac{k^2 m_1 M_1}{\nu_1} + \frac{k^2 m_2 M_2}{\nu_2}, \tag{10}$$

where $\nu_i^2 = \lambda_i^2 - \mu_i^2$. Here, λ_i and μ_i are elliptic coordinates in the system of ellipses having the same foci as the ellipses of P_1 and P_2, respectively, and

$$M_i = \frac{2}{\pi}K(\chi_i), \quad i=1,2. \tag{11}$$

$K(\chi_i)$ represents the elliptic integral of the first kind:

$$K(\chi_i) = \int_0^{\pi/2}(1-\chi_i^2\sin^2\varphi)^{-1/2}\mathrm{d}\varphi, \tag{12}$$

with χ_i given by:

$$\chi_i^2 = \frac{4a_i^2 e^2 - \mu_i^2}{\lambda_i^2-\mu_i^2}, \quad i=1,2. \tag{13}$$

The transformation between (λ_i,μ_i), $i=1,2$, and the cartesian coordinates $(\overline{\xi},\overline{\eta})$ is given by (Pál (1982), Pál (1983), Rein (1940)):

$$\begin{aligned} \lambda_i^2 &= \tfrac{f_i+(f_i^2-g_i)^{1/2}}{2}, \\ \mu_i^2 &= \tfrac{f_i-(f_i^2-g_i)^{1/2}}{2},\ i=1,2, \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_i &= (\overline{\xi}-(-1)^i a_i)^2 + \overline{\eta_i}^2 + 3a_i^2 e^2, \\ g_i &= 12 a_i^2 e^2 \overline{\eta}^2,\ i=1,2. \end{aligned} \tag{15}$$

Rein's trajectories for the relative motion of P can be obtained from the differential equations of motion:

$$\begin{aligned} \ddot{\overline{\xi}} - 2n\dot{\overline{\eta}} &= \overline{U}_{\overline{\xi}}, \\ \ddot{\overline{\eta}} + 2n\dot{\overline{\xi}} &= \overline{U}_{\overline{\eta}}. \end{aligned} \tag{16}$$

Here we can see the advantage of Rein's method: the force function $\overline{U}$ is not an explicit function of time anymore and the equations of motion (16), admit a first integral analogous to Jacobi's first integral:

$$\dot{\overline{\xi}}^2 + \dot{\overline{\eta}}^2 = 2(\overline{U} + h), \tag{17}$$

where h is the constant of integration.

Additionally, zero relative velocity curves are given by the equation $\overline{U}(\overline{\xi}, \overline{\eta}) + h = 0$ (Pál & Oproiu (1991)).

3. PERTURBATION EFFECT OF DRAG

The equations of the planar motion of the third body, perturbed by an arbitrary external force $\mathbf{F}(F_\xi, F_\eta)$, are given by Murray (1994), Murray & Dermott (1999):

$$\begin{aligned} \ddot{\overline{\xi}} - 2n\dot{\overline{\eta}} &= \overline{U}_{\overline{\xi}} + F_{\overline{\xi}}, \\ \ddot{\overline{\eta}} + 2n\dot{\overline{\xi}} &= \overline{U}_{\overline{\eta}} + F_{\overline{\eta}}. \end{aligned} \tag{18}$$

Generally, the components of $\mathbf{F}$ are functions of the position $(\overline{\xi}, \overline{\eta})$ and the velocity $(\dot{\overline{\xi}}, \dot{\overline{\eta}})$. For our problem, we will also consider $\mathbf{F} = O(K)$, where K is the drag parameter. Note that when $\mathbf{F} = \mathbf{0}$, we have Rein's problem.

From Eq.(18), we have:

$$\dot{\overline{\xi}}\,\ddot{\overline{\xi}} + \dot{\overline{\eta}}\,\ddot{\overline{\eta}} - (\dot{\overline{\xi}}\,\overline{U}_{\overline{\xi}} + \dot{\overline{\eta}}\,\overline{U}_{\overline{\eta}}) = \dot{\overline{\xi}}\,F_{\overline{\xi}} + \dot{\overline{\eta}}\,F_{\overline{\eta}}, \tag{19}$$

If we denote $C_J = 2\,\overline{U} - \dot{\overline{\xi}}^2 - \dot{\overline{\eta}}^2$, we get

$$\frac{\mathrm{d}C_J}{\mathrm{d}t} = -2\,(\dot{\overline{\xi}}\,F_{\overline{\xi}} + \dot{\overline{\eta}}\,F_{\overline{\eta}}) \tag{20}$$

Above, $(\dot{\overline{\xi}}\,F_{\overline{\xi}} + \dot{\overline{\eta}}\,F_{\overline{\eta}})$ is the work done by the drag force $\mathbf{F}$, per unit of time.

In what follows we'll consider that the third body (mass point) in Rein's problem is moving in a resisting medium with a relative velocity equal to the relative velocity of the rotating medium. In addition, we consider the drag force to be proportional to the square of the velocity v of the particle in the rotating frame: $\mathbf{F} = Kv\mathbf{v} = K(v\dot{\overline{\xi}}, v\dot{\overline{\eta}})$, where $K < 0$ is the drag constant; this force is called *quadratic drag*. The equations of the planar motion of P are:

$$\begin{aligned} \ddot{\overline{\xi}} - 2n\dot{\overline{\eta}} &= \overline{U}_{\overline{\xi}} + Kv\dot{\overline{\xi}}, \\ \ddot{\overline{\eta}} + 2n\dot{\overline{\xi}} &= \overline{U}_{\overline{\eta}} + Kv\dot{\overline{\eta}}, \end{aligned} \tag{21}$$

and from (20) we have:

$$\frac{\mathrm{d}C_J}{\mathrm{d}t} = -2\,K\,v\,(\dot{\overline{\xi}}^2 + \dot{\overline{\eta}}^2)\ >\ 0 \tag{22}$$

4. APPLICATION TO THE EARTH-MOON SYSTEM. ORBITS AROUND L_4

In this section we consider the Earth-Moon system and with the notations introduced in previous sections. Let P be a spacecraft, P_1 the Earth and P_2 the Moon.

For this application, units are taken such that both the sum of the masses of P_1 and P_2 and the distance between the points equal 1. Moreover, the unit of time is taken such that the time period of P_1 and P_2 around the center of mass equals 2π units. This way $n = 1$ and $k^2 = 1$. For the Earth-Moon system, the eccentricity is e = 0.0549 and the Moon to Earth ratio is $m_2\,/m_1$ = 1/81.45.

Equilibrium points ($\dot{\overline{\xi}} = \dot{\overline{\eta}} = \ddot{\overline{\xi}} = \dot{\overline{\eta}} = 0$) of the system (18) are given by the system of equations:

$$\begin{aligned} \overline{U}_{\overline{\xi}} + F_{\overline{\xi}} = 0, \\ \overline{U}_{\overline{\eta}} + F_{\overline{\eta}} = 0. \end{aligned} \tag{23}$$

If $\mathbf{F} = \mathbf{0}$, the algebraic system of equations (23) has five roots, represented by three collinear points $\overline{L}_i(\overline{x}_i, 0)$, $i = 1, 2, 3$, and two quasi-equilateral points $\overline{L}_i(\overline{x}_i, \overline{y}_i)$, $i = 4, 5$.

Using Rein's averaged potential for the Earth-Moon system we determined the coordinates of these points (see also Barbosu & Oproiu (2009), Barbosu & Oproiu (2013), Pál & Oproiu (1988), Pál, Oproiu, & Macaria (1990)).

The coordinates of the collinear points are:

$\overline{L}_1(0.8434404371, 0)$,

$\overline{L}_2(1.1486457454, 0)$,

$\overline{L}_3(-1.0050521268, 0)$.

The coordinates of the quasi-equilateral points are:

$\overline{L}_4(0.4827669364, 0.8689222737)$,

$\overline{L}_5(0.4827669364, -0.8689222737)$.

Note that the drag of the uniformly rotating medium does not affect the position of equilibrium points (SHU Si-hui *et al.* (2005)).

Let us now consider an application around the equilibrium point $\overline{L}_4$, with a drag constant $K = -0.02$ and the following initial conditions for the equations of motion (21):

$$\overline{\xi}(0) = 0.4827669364,\ \dot{\overline{\xi}}(0) = 0,$$

$$\overline{\eta}(0) = 0.8689222737,\ \dot{\overline{\eta}}(0) = -0.02.$$

Integrating numerically using RK4 we obtained the trajectory given in Figure 1.

Note that the point P had a non zero initial velocity and in Figure 1 we saw what we expected, the drag force causing instability and the point P moving away from the L_4 equilibrium point.

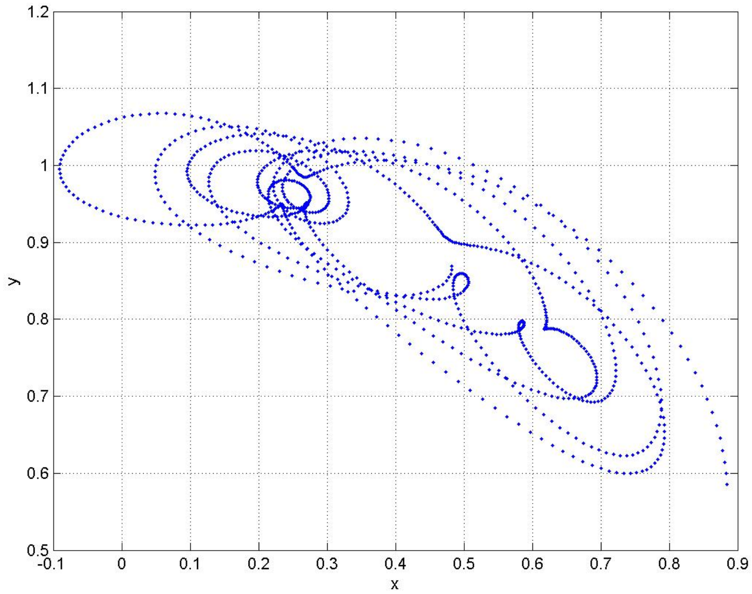

Fig. 1 – Trajectory of P in the vicinity of L_4, when $m_2/m_1 = 1/81.45$ and the drag constant is $K = -0.02$.

On another hand, for initial conditions:
$\overline{\xi}(0) = 0.4827669364$, $\dot{\overline{\xi}}(0) = 0$,

$\overline{\eta}(0) = 0.8689222737$, $\dot{\overline{\eta}}(0) = 0$.
we obtain the results given in Figure 2, where the motion is stable around L_4. In this case, the point P had zero initial velocity.

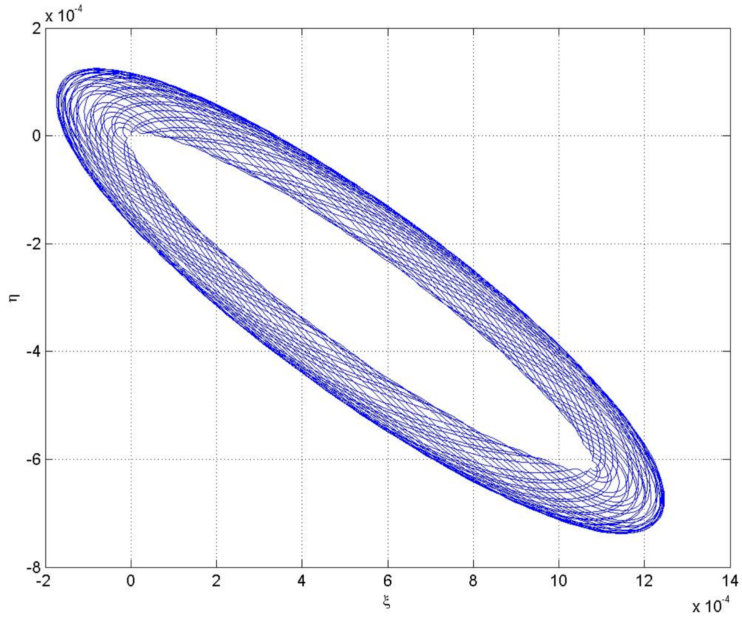

Fig. 2 – Trajectory around of L_4 when m_2/m_1 = 1/ 81.45 and drag constant K = -0.05.

Rein's scheme for the ER3BP presents a real interest because of the expression of the averaged force function $\overline{U}$, that does not explicitly depend on t, and the first integral (17), that provides an effective tool for qualitative studies. In addition, for small values of the eccentricity, Rein's model is a very good approximation of the ER3BP and it can further be generalized and used in other applications.

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