STABILITY IN SENSE OF LYAPUNOV OF CIRCULAR ORBITS IN MANEV POTENTIAL

CRISTINA BLAGA

Faculty of Mathematics and Computer Science
Babeş-Bolyai University,
Kogălniceanu Street, 1, Cluj-Napoca, Romania
Email: cpblaga@math.ubbcluj.ro

Abstract. In this article we consider the motion of two bodies under the action of a Manev central force. We obtain the radius of the circular orbit and analyze its stability in sense of Lyapunov. Drawn on the first integrals of angular momentum and energy, we build a positive definite function which satisfies the Lyapunov's theorem of stability. The existence of the Lyapunov function prove that the circular orbits in Manev two body problem are stable at any perturbation. In the end we compare these results with those valid for the circular orbits in the Newtonian gravitational field.

Key words: celestial mechanics – space sciences.

1. INTRODUCTION

In the nineteen twenties, Georgi Manev (Maneff in German and French spelling) published several papers (Maneff (1925), Maneff (1929), Maneff (1930)), in which he proposed a nonrelativistic gravitational law, able to explain certain dynamical phenomena observed in the solar system, not explained in the frame of classical mechanics. Manev obtained his model as a consequence of Max Planck's action-reaction principle in Maneff (1925) and considered it as a substitute to general relativity. In his papers he noticed that this model provides a good explanation to the observed motion in the solar system. Manev emphasized that under a particle of mass m_2, which is moving in the static gravitational field due to mass m_1 ($m_1 > m_2$), acts a central force of the form

$$F(r) = -\frac{Gm_1m_2}{r^2}\left(1 + \frac{3G(m_1+m_2)}{c^2r}\right) \tag{1}$$

where r is the distance from the particle m_2 to the center of mass of m_1, G the Newtonian gravitational constant and c the velocity of light. The force (1) - known as Manev force - differs from the Newtonian force through the additional term inverse proportional with r^3. Under the action of Manev force the mass m_2 describes a precessional ellipse, with m_1, assimilated to a point mass, placed in one focus of the ellipse. Around the point in which m_1 is situated, the apsidal line is rotating and so perihelion advance of Mercury could be qualitatively explained. Sir Isaac Newton

considered a similar force in his book *Philosophiae Naturalis Principia Mathematica* issued in 1687. He proved that if to the Newtonian gravitational force is added the term μ/r^3 (where $\mu > 0$ and r the distance between the particles), the particle describes a precessional ellipse (see for example Newton (1999), Book I, Section IX, Proposition XLIV, Theorem XIV). At that time, Newton was looking for an explanation to the observed motion of the perigee of Moon. There were other attempts to use forces similar to Manev force to explain certain dynamical phenomena observed in the solar system, unexplained in the classical mechanics. A comprehensive list of such historical works is given in Diacu *et al.* (1995).

After several decades in which Manev model was forgotten, it was brought to light by Diacu (1993). In nineteen nineties the motion under Manev force was analyzed in order to model different phenomena from celestial mechanics, stellar dynamics or from astrophysics. A survey of the papers devoted to the applications of Manev gravitational field could be found in Haranas and Mioc (2009).

The investigation of the motion in Manev's gravitational field revealed that in the solar system it gives a theoretical approximation as good as general relativity, building a bridge between classical mechanics and general relativity. An investigation of the ways in which modified Manev potential could model certain general relativistic results in the frame of Newtonian mechanics was done by Ivanov and Prodanov (2005).They analyzed the circular orbits around a rotating and non-rotating massive body or in a parametrized post-newtonian metric.

The two body problem in Manev potential was considered by several authors. The analytic solution of the problem and its behavior near collision was investigated by Diacu *et al.* (1995). The solution of the regularized equations of motion for different initial conditions and hypothesis about regularization was obtained by Mioc and Stoica (1995a) and Mioc and Stoica (1995b).

In this paper we consider the circular orbits in the two-body problem in Manev potential. In space dynamics the circular orbits are met often. The spacecrafts exploring the Universe are moving sometimes on circular orbits and also the transfer between the bodies from the solar system is done sometimes on segments of circular paths (Roy, 2004). The geostationary satellites have also near circular orbits.

We study here the stability in the sense of Lyapunov of the circular orbits in Manev potential. In the section 2 we obtain the equations of motion of a body under the action of Manev central force (1) and determine the radius of the circular orbit. In section 3, we derive the Lagrangian and obtain the first integrals of motion. Drawn on the angular momentum and energy integral, we build a positive defined function which fulfills the conditions from the Lyapunov's stability theorem. Using this function on the ground of the Lyapunov's theorem of stability we reach the conclusion that the circular orbit is stable at any perturbation. In the last section we compare characteristics of the circular orbit in Manev potential with those from the

Newtonian gravitational field.

2. CIRCULAR ORBITS IN MANEV'S POTENTIAL

2.1. EQUATIONS OF MOTION

Let us consider the motion of two interacting bodies of masses m_1 and m_2 ($m_1 > m_2$) under a Manev force (1). We assume that bodies are assimilated to point masses located in their center of mass and the Manev force is an attracting force, acting in the line joining the bodies. If the position vectors of the bodies are $\vec{r_1}$ and $\vec{r_2}$ respectively, the equations of motion are:

$$m_1\ddot{\vec{r}}_1 = \vec{F}_{12} \tag{2}$$

$$m_2\ddot{\vec{r}}_2 = \vec{F}_{21} \tag{3}$$

where

$$\vec{F}_{12} = -\vec{F}_{21} = -F\frac{\vec{r}}{r}$$

and $\vec{r} = \vec{r_2} - \vec{r_1}$ is the position vector of m_2 relative to m_1. After several straight-forward transformations we get the equation of relative motion of m_2 in respect with m_1

$$\mu\ddot{\vec{r}} = F(r)\frac{\vec{r}}{r}, \tag{4}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

During motion, the force acting on m_2 is directed to the center of mass of m_1, it is a central force. It depends only on the radial coordinate r, therefore we can determine a scalar function $V(r)$ from which Manev force is derived ($F = -dV(r)/dr$). This function known as Manev potential and is given by

$$V(r) = -\frac{Gm_1 m_2}{r}\left(1 + \frac{3G(m_1 + m_2)}{2c^2 r}\right), \tag{5}$$

where quantities m_1, m_2, G, c and r have the same meaning as in (1).

If we multiply vectorial the equation of relative motion (4) with position vector $\vec{r}$, we get the angular momentum integral

$$\vec{r} \times \mu\dot{\vec{r}} = \vec{C} \tag{6}$$

where the vector $\vec{C}$ is constant, perpendicular on the plane in which the motion took place. Like in the Newtonian case, the motion is restricted to the plane determined by the initial position of m_1, the initial position vector of m_2 relative to m_1 and its initial velocity in respect to m_1.

If we multiply scalar (4) with the relative velocity of m_2 in respect with m_1, $\vec{v} = \dot{\vec{r}}$, after some algebra we get the energy integral

$$\frac{\mu}{2}v^2 + V(r) = h \tag{7}$$

where v is the length of velocity and h the total energy. The relation (7) stats that the total energy is conserved during the motion.

The motion of m_2 is planar, thus further we will analyze the motion in the orbital plane. In polar coordinates (r,φ) the acceleration of m_2 is

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\vec{e}_\varphi \,, \tag{8}$$

where $\vec{e}_r$ and $\vec{e}_\varphi$ are the unit vectors on the radial and polar axis respectively. After some straightforward algebra, the projections of the equation of relative motion (4) on the radial and polar axis become:

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\gamma}{r^2}\left(1 + \frac{3\gamma}{c^2 r}\right) \tag{9}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}) = 0 \tag{10}$$

where $\gamma = G(m_1 + m_2)$ is the gravitational parameter. From (10) $r^2\dot{\varphi} = C$, with C a constant. The relation (10) is equivalent with integral of angular momentum (6), because the length of angular momentum is $|\vec{C}| = \mu r^2\dot{\varphi}$.

2.2. EXISTENCE OF CIRCULAR ORBITS

In polar coordinates, the circular orbits are given by $r = r_0 = $ constant, where the constant r_0 is the radius of the circular orbit. From (10) $\dot{\varphi} = C/r_0^2$, the angular velocity is constant and the circular motion is uniform, like in the Newtonian case. On a circular orbit $\dot{r} = \ddot{r} = 0$ and from (9) we get

$$r_0 = \frac{C^2}{\gamma}\left(1 - \frac{3\gamma^2}{C^2 c^2}\right). \tag{11}$$

We observe that if $C^2 > 3\gamma^2/c^2$, then the positive real number r_0 given by (11) is the radius of the circular orbit in the Manev two-body problem. In the classical Newtonian two-body problem the radius of the circular orbit is $r_0 = C^2/\gamma$ (Goldstein, 1980). Comparing the radius of the circular orbit in the Manev and Newtonian potential, for a given angular momentum, we note that in the Manev case the radius of the circular orbit is smaller then in the Newtonian case. For a given angular momentum constant C, the difference between the radius of the circle described in the Manev potential (11) and the corresponding circular orbit from the Kepler problem is

$$\Delta r_0 = r_{0N} - r_{0M} = \frac{3\gamma^2}{c^2}, \tag{12}$$

where r_{0N} and r_{0M} are the radius of the circular orbits in Newtonian and Manev potential, respectively. The difference (12) does not depend on C, although the radius of the circular orbits are dependent on C. It depends on the ratio between the gravitational parameter and the speed of light.

3. STABILITY OF THE CIRCULAR ORBIT

Let us consider the stability of the circular orbit in Manev two-body problem in respect with the variables r, θ, φ and their time derivatives $\dot{r}$, $\dot{\theta}$, $\dot{\varphi}$, where (r, θ, φ) are the spherical coordinates, r is the radial coordinate, θ the latitude and φ the longitude. We assume that the circular orbit lies in the equatorial plane.

The Lagrangian for the Manev two-body is

$$L = T - V(r) \tag{13}$$

where the kinetic energy T in spherical coordinates is

$$T = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos\theta\dot{\varphi}^2\right) \tag{14}$$

and the potential $V(r)$ from (5) in terms of the reduced mass μ and the gravitational parameter γ is

$$V(r) = -\frac{\mu\gamma}{r}\left(1 + \frac{3\gamma}{2c^2 r}\right). \tag{15}$$

The coordinate φ does not appear explicitly in the Lagrangian (13), it is a cyclic coordinate. Thus, from the Lagrange equation for φ we get the angular momentum integral

$$r^2\cos^2\theta\dot{\varphi} = b, \tag{16}$$

with b a constant.

The perturbed motion is given by x_i, $i = \overline{1,5}$, with

$$r = r_0 + x_1, \quad \dot{r} = x_2, \quad \theta = x_3, \quad \dot{\theta} = x_4, \quad \dot{\varphi} = \dot{\varphi}_0 + x_5, \tag{17}$$

where $\dot{\varphi}_0 = C/r_0^2$ and r_0 is given by (11).

According to the stability theorem in Lyapunov sense, we are looking for a function $\mathcal{F}(\mathbf{x})$, with $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, which is positive definite in a vicinity of the unperturbed motion $\mathbf{x} = \mathbf{0}$ (Hahn, 1967). Usually, the first integrals of motion are used to build the Lyapunov function. Therefore, we consider the energy integral (7), divided by $\mu/2$, for the perturbed motion

$$\begin{aligned}
F_1(\mathbf{x}) &= x_2^2 + (r_0 + x_1)^2 x_4^2 + (r_0 + x_1)^2(\dot{\varphi} + x_5)^2\cos^2 x_3 - \tag{18}\\
&- \frac{2\gamma}{r_0 + x_1} - \frac{3\gamma^2}{c^2(r_0 + x_1)^2}
\end{aligned}$$

and the integral of motion (16) for the perturbed motion

$$F_2(\mathbf{x}) = (r_0 + x_1)^2 (\dot{\varphi} + x_5) \cos^2 x_3 \,. \tag{19}$$

These functions do not have definite sign in respect with x_i, $i = \overline{1,5}$, thus we consider the function

$$\mathcal{F}(\mathbf{x}) = F_1(\mathbf{x}) - F_1(\mathbf{0}) + \lambda[F_2(\mathbf{x}) - F_2(\mathbf{0})] + \nu[F_2^2(\mathbf{x}) - F_2^2(\mathbf{0})], \tag{20}$$

where $\mathbf{0} = (r_0, \varphi_0, 0, \dot{\varphi}_0, 0)$ denotes the circular unperturbed motion and λ, ν are two constants. Further, we examine whether there are real numbers λ and ν, so that $\mathcal{F}(\mathbf{x})$ is a positive definite function in a neighborhood of the circular unperturbed motion, given by $\mathbf{0} = (r_0, \varphi_0, 0, \dot{\varphi}_0, 0)$.

Developing in Taylor series $\cos^2 x_3$ in the vicinity of 0 and $2\gamma/r^2 + 3\gamma^2/(c^2 r^2)$ in the neighborhood of r_0, neglecting the terms of order 3 and higher and keeping in mind that from (9) for a circular orbit

$$r_0 \dot{\varphi}_0^2 = \frac{\gamma}{r_0^2} \left(1 + \frac{3\gamma}{c^2 r_0} \right) \tag{21}$$

we get

$$\begin{aligned} F_1(\mathbf{x}) - F_1(\mathbf{0}) &= 4 r_0 \dot{\varphi}_0^2 x_1 + 2 r_0^2 \dot{\varphi}_0 x_5 - \left(\dot{\varphi}_0^2 + \frac{3\gamma^2}{c^2 r_0^4} \right) x_1^2 + x_2^2 - \tag{22} \\ &\quad - r_0^2 \dot{\varphi}_0^2 x_3^2 + r_0 \left(x_4^2 + x_5^2 \right) + 4 r_0 \dot{\varphi}_0 x_1 x_5 \end{aligned}$$

$$F_2(\mathbf{x}) - F_2(\mathbf{0}) = 2 r_0 \dot{\varphi}_0 x_1 + r_0^2 x_5 + \dot{\varphi}_0 x_1^2 - r_0^2 \dot{\varphi}_0 x_3^2 + 2 r_0 x_1 x_5 \tag{23}$$

and

$$\begin{aligned} F_2(\mathbf{x})^2 - F_2(\mathbf{0})^2 &= 4 r_0^3 \dot{\varphi}_0^2 x_1 + 2 r_0^4 \dot{\varphi}_0 x_5 + 6 r_0^2 \dot{\varphi}_0^2 x_1^2 - 2 r_0^4 \dot{\varphi}_0^2 x_3^2 + \tag{24} \\ &\quad + 8 r_0^3 \dot{\varphi}_0 x_1 x_5 + r_0^4 x_5^2 \,. \end{aligned}$$

After replacing (22), (23) and (24) in (20), we obtain that the necessary and sufficient condition for $\mathcal{F}$ to have an extremum in $x_i = 0$, $i = \overline{1,5}$ is

$$\lambda = -2\dot{\varphi}_0 \left(1 + r_0^2 \nu \right) \,. \tag{25}$$

We substitute λ from (25) in (20) and write $\mathcal{F}(\mathbf{x})$ as a sum of two functions

$$\mathcal{F}(\mathbf{x}) = \mathcal{F}_1(x_2, x_3, x_4) + \mathcal{F}_2(x_1, x_5) \tag{26}$$

with

$$\mathcal{F}_1(x_2, x_3, x_4) = x_2^2 + r_0^2 \dot{\varphi}^2 x_3^2 + r_0^2 x_4^2 \tag{27}$$

a positive definite function in respect with x_2, x_3, x_4 and

$$\mathcal{F}_2(x_1, x_5) = c_{11} x_1^2 + 2 c_{12} x_1 x_5 + c_{22} x_5 \,, \tag{28}$$

with

$$c_{11} = 4\dot{\varphi}_0^2 \left(r_0^2 \nu - 1\right) + \frac{\gamma}{r_0^3}, \quad c_{12} = c_{21} = 2r_0^3 \dot{\varphi}_0 \nu, \quad c_{22} = r_0^2 \left(1 + \nu r_0^2\right), \quad (29)$$

a quadratic form in x_1 and x_5. $\mathcal{F}_2(x_1, x_5)$ is positive definite if and only if all principal minors are positive. In other words, if $c_{11} > 0$ and $d = c_{11}c_{22} - c_{12}^2 > 0$. After some algebra, using (21), these two inequalities lead us to

$$\nu > \max \left\{ \frac{3\gamma \left(c^2 r_0 + 4\gamma\right)}{c^2 r_0^6 \dot{\varphi}_0^2}, \frac{3 \left(c^2 r_0 + 4\gamma\right)}{r_0^3 c^2} \right\}. \quad (30)$$

It is easy to check that (30) is fulfilled if $\nu > 3(c^2 r_0 + 4\gamma)/(r_0^3 c^2)$. Thus $\mathcal{F}_2(x_1, x_5)$ is a positive definite function if ν satisfies (30). Further, $\mathcal{F}(\mathbf{x})$ is a positive definite function, therefore the circular orbit (11) is stable in the sense of Lyapunov. This result generalize the Lyapunov stability of circular orbits in Newtonian gravitational field (Dragoş (1976), Goldstein (1980)).

4. CONCLUSIONS

The aim of the study was to examine the stability in the sense of Lyapunov of circular orbits in Manev two body problem. The central force acting on the bodies differs from the Newtonian force through the term inverse proportional with r^3, therefore further we will compare the properties of the circular motion in Manev and Newtonian two body problem.

In Manev's case we obtained that the radius of the circular orbit described by m_2, $(m_1 > m_2)$, depends on the angular momentum constant C, like in the Newtonian case. For a given C, the circle described in Manev's potential has a smaller radius. The quantity with which the radius of the circular orbit from the Manev case is smaller than the radius from the Newtonian case does not depend on C.

From $C = r_0^2 \dot{\varphi}_0$ we get that the angular velocity on the circular orbit is constant, the motion is uniform. Using the relation between the radius of the circular orbit in Manev and Newtonian case, we get that for a given C, between the corresponding angular velocities exists the inequality $\dot{\varphi}_{0M} = C/r_{0M}^2 > C/r_{0N}^2 = \dot{\varphi}_{0N}$. In other words, for a given angular momentum, the secondary body in Manev two body problem has a higher angular velocity then in Kepler problem. In addition, the period of the circular motion in Manev two body problem is smaller then in Newtonian case, because $T_M = 2\pi/\dot{\varphi}_{0M} < 2\pi/\dot{\varphi}_{0N} = T_N$.

In table (1) we gathered the results about the stability in sense of Lyapunov of circular orbit in Manev and Newtonian two-body problem. The radius of the orbits in these two cases are different for a given C, but the analytical expression for the relation between the unknown constants ν and λ is the same. Even so, for a given

Table 1

Circular orbits in Manev and Newtonian potential. The properties of the circular orbit in the
Newtonian potential are from Dragoş (1976).

	Manev	Newton
Radius of circular orbit (r_0)	$\frac{C^2}{\gamma}\left(1 - \frac{3\gamma^2}{C^2 c^2}\right)$	$\frac{C^2}{\gamma}$
Relation between ν and λ	$\lambda = -2\dot{\varphi}_0\left(1 + r_0^2\nu\right)$	$\lambda = -2\dot{\varphi}_0\left(1 + r_0^2\nu\right)$
Necessary and sufficient condition for the stability of the circular orbit	$\nu > \frac{3\left(c^2 r_0 + 4\gamma\right)}{c^2 r_0^3}$	$\nu > \frac{3}{r_0^2}$

value of angular momentum C, we can obtain different values for λ and ν, which
assures the stability of the circular orbit. The necessary and sufficient condition for
the Lyapunov stability of the circular orbit in Manev potential is reduced to that from
the two-body problem in Newtonian potential. Therefore we can conclude, that the
results from Manev two-body problem generalize those from the classical Newtonian
case. The addition of the term inverse proportional with the third power of the distance between the bodies let us explain certain dynamical phenomena unexplained
in the Newtonian gravitational field, but it does not destroy the Lyapunov stability of
the circular orbit.

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