SYMMETRY AND REDUCTIONS OF INTEGRABLE DYNAMICAL SYSTEMS: PEAKON AND THE TODA CHAIN SYSTEMS

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Abstract. We are analyzing several types of dynamical systems which are both integrable and important for physical applications. The first type are the so-called peakon systems that appear in the singular solutions of the Camassa-Holm equation describing special types of water waves. The second type are Toda chain systems, that describe molecule interactions. Their complexifications model soliton interactions in the adiabatic approximation. We analyze the algebraic aspects of the Toda chains and describe their real Hamiltonian forms.

Key words: integrable dynamical systems, peakons, Toda lattices.

1. INTRODUCTION

Many integrable dynamical systems possess various symmetries, expressed via invariant actions of certain finite groups transforming their variables. In the simplest case these are involutions, exchanging for example two sets of variables. The symmetry groups allow also two or more variables to be made equal (or dependent), thus reducing the overall number of the independent variables and simplifying the system and their solution. The reduced system of course inherits some of the properties of the original system.

Most often these integrable systems possess Lax representation $L_t = [L, M]$; the operators L and M usually take values in some underlying simple Lie algebra $\mathfrak{g}$. The Lax operators of the reduced systems may take values in an subalgebra of $\mathfrak{g}$ which is invariant under a finite order automorphism of $\mathfrak{g}$ (Mikhailov, 1981).

An approach for obtaining new families of such systems is based on the construction of their real Hamiltonian forms (RHF) (Gerdjikov *et al.* , 2002, 2004). This approach first complexifies the dynamical system and then picks up the corresponding RHF by applying a Cartan involution. This construction is specially transparent for systems that possess Lax pairs.

In Sections 2 and 3 we study a special involution of the so-called peakon systems. Section 4 analyzes the construction of RHF of Toda chain models related to affine twisted Kac-Moody algebras (Helgasson, 2001).

2. THE CAMASSA-HOLM EQUATION AND THE PEAKON SYSTEM

The Camassa-Holm (CH) equation (Camassa *et al.* , 1993) among his many and rich properties and structures is famous for possessing the so-called *peakon* solutions. These are solilary waves with a shape, given by the Green function $G(x)$ of the Helmholtz operator $1-\partial_x^2$, i.e.

$$(1-\partial_x^2)G(x)=\delta(x) \tag{1}$$

The solution with decaying properties is $G(x)=\frac{1}{2}\exp(-|x|)$. The CH equation has the form (all variables are real)

$$m_t+2mu_x+m_xu=0, \qquad m=u-u_{xx}, \tag{2}$$

and is integrable due to the Lax pair

$$\Psi_{xx}=\Big(\frac{1}{4}+\lambda m\Big)\Psi, \qquad \Psi_t=\Big(\frac{1}{2\lambda}-u\Big)\Psi_x+\frac{u_x}{2}\Psi, \tag{3}$$

The N-peakon solution is

$$u(x,t)=\sum_{k=1}^{N}p_k(t)G(x-x_k(t)), \qquad m(x,t)=\sum_{k=1}^{N}p_k(t)\delta(x-q_k(t)). \tag{4}$$

The peakon parameters $p_k(t),q_k(t)$ satisfy an integrable Hamiltonian system

$$\dot{q}_k=\frac{\partial H_N}{\partial p_k} \qquad \dot{p}_k=-\frac{\partial H_N}{\partial q_k}, \tag{5}$$

where the Hamiltonian is given by

$$H_N=\frac{1}{2}\sum_{k=1}^{N}p_ip_jG(q_i-q_j). \tag{6}$$

The Lax representation $\dot{L}=[L,M]$ of (5) is:

$$L_{jk}=\sqrt{p_jp_k}\alpha(q_j-q_k), \qquad M_{jk}=-2\kappa\sqrt{p_jp_k}\alpha'(q_j-q_k), \tag{7}$$

Here $\alpha'(x)$ denotes derivative with respect to the argument of the function $\alpha(x)=2G(x/2)$. The peakon Hamiltonian H_N may be expressed as a function of the invariants of the matrix L as

$$H_N=-\mathrm{tr}\ L^2+2(\mathrm{tr}\ L)^2. \tag{8}$$

The fact that $G(x)$ is an even function leads to the following properties:

(i) The N coordinates q_j keep their initial ordering with time. Hence, we can assume *natural ordering* $q_1 \leq q_2 \leq \ldots \leq q_N$. Physically this means that the particles do not cross each other;

(ii) The conjugated momenta p_j keep their initial sign. Thus, there are no ambiguities in the expressions $\sqrt{p_i p_j}$.

The 'peakon-antipeakon' case can be easily solved, and the solution is

$$u(x,t) = p(G(x+q/2) - G(x-q/2)), \qquad m(x,t) = p(\delta(x+q/2) - \delta(x-q/2)), \tag{9}$$

where

$$q = -\log \operatorname{sech}^2(ct) \qquad p = \pm \frac{2c}{\tanh(ct)}. \tag{10}$$

Note that this is an odd-function solution of the CH equation. More details can be found e.g. in (Holm *et al.* , 2009, 2010; Ragnisco *et al.* , 1996; Beals *et al.* , 1999; Holm *et al.* , 2005). It is well known that geometrically the CH equation is a geodesic flow equation on the diffeomorphism group of the circle, (Holm *et al.* , 1998, 2009; Constantin *et al.* , 2003).

It is known that there are more-general odd-function CH solutions (Constantin, 2000), and thus there must exist a multi-peakon odd-function solution with

$$p_k = -p_{\bar{k}}, \qquad q_k = -q_{\bar{k}}, \tag{11}$$

where $\bar{k} = N+1-k$, see the next section. This is related to the presence of a symmetry, more specifically a $\mathbb{Z}_2$- automorphism. Let us assume that $N = 2n$ and the variables are divided into two groups, positive and negative as follows $q_1 < q_2 < \ldots < q_n < 0 < q_{n+1} < \ldots < q_{2n}$; $p_1 < p_2 < \ldots < p_n < 0 < p_{n+1} < \ldots < p_{2n}$. The action of the $\mathbb{Z}_2$- automorphism is

$$p_k \to -p_{\bar{k}}, \qquad q_k \to -q_{\bar{k}}, \tag{12}$$

which exchanges the two groups of variables.

3. PEAKON SYSTEM WITH $\mathbb{Z}_2$ SYMMETRY AND RELATION TO THE TODA CHAIN

The Lax pair of the peakon system takes values in the algebra $\mathfrak{g} \simeq sl(N)$. There is a $\mathbb{Z}_2$ - automorphism, given by (12). This allows the following reduction of the peakon system. The structure of L and M under the invariance with respect to the $\mathbb{Z}_2$- involution $p_k = -p_{\bar{k}}$, $q_k = -q_{\bar{k}}$ where $\bar{k} = N+1-k$, and thus $N = 2n$ is:

$$L = \begin{pmatrix} L_1 & iA \\ iA^T & -SL_1S \end{pmatrix} \quad M = \begin{pmatrix} M_1 & iA \\ -iA^T & SM_1S \end{pmatrix}, \quad S = \sum_{k=1}^{n} E_{k,n+1-k}, \tag{13}$$

where E_{km} is $n\times n$ matrix $(E_{km})_{sp} = \delta_{ks}\delta_{mp}$; L and M are $2n\times 2n$ matrices; L is symmetric and M is anti-symmetric. This means that their $n\times n$ blocks satisfy:

$$L_1 = L_1^T, \qquad M_1 = -M_1^T, \qquad A^T = -SAS. \tag{14}$$

The particular entries of A, M_1 and L_1 can be recovered from (7). The equations that follow from $\dot{L} = [L, M]$ are

$$\dot{L_1} = [L_1, M_1] + 2AA^T, \qquad \dot{A} = (L_1 - M_1)A + AS(M_1 + L_1)S. \tag{15}$$

Finally, one can verify that L and M satisfy the involutions

$$\Sigma L \Sigma^{-1} = -L^T, \qquad \Sigma M \Sigma^{-1} = -M^T, \qquad \Sigma = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}. \tag{16}$$

This is a reduction of $sl(2n)$ to the symplectic algebra $sp(n)$ and provides odd-function solutions of the CH equation.

In (Ragnisco *et al.* , 1996) it is proven the interesting fact that if p_k, q_k evolve according to the Hamiltonian H_N, the 'Flashka variables'

$$A_j = -\frac{1}{p_j p_{j+1}} \frac{e_j}{1-e_j^2}, \qquad B_j = \frac{1}{p_j} \frac{1-e_{j-1}^2 e_j^2}{(1-e_{j-1}^2)(1-e_j^2)}, \qquad e_j = e^{\pm(q_{j+1}-q_j)/2}, \tag{17}$$

evolve according to the Hamiltonian $H = \ln\det\Lambda$, where Λ is the tri-diagonal matrix:

$$\Lambda = \sum_{k=1}^{n} B_k E_{kk} + \sum_{k=1}^{n-1} A_k (E_{k,k+1} + E_{k+1,k}). \tag{18}$$

Thus, the peakon lattice is one of the commuting flows of the TC hierarchy (Toda, 1967, 1989; Manakov, 1974; Flaschka, 1974). For $N = 2n$ with eq. (11) we have

$$A_k = A_{N-k}, \qquad B_k = -B_{N-k+1}, \tag{19}$$

i.e. Λ reduces from $sl(2n)$ to its subalgebra $sp(2n)$, see e.g. (Gerdjikov *et al.* , 1998).

4. THE TODA CHAINS AND THEIR REAL HAMILTONIAN FORMS

It is well known that to each simple Lie algebra one can relate a Toda chain (TC) system. These systems are known to describe molecule interactions (Toda, 1967). There are two main classes of Toda chains: i) the conformal ones (Toda, 1967) and ii) the affine ones (Olshanetzky *et al.* , 1983; Bogoyavlensky, 2006).

In the 1990-ies it was discovered that the complexifications of the conformal TC related to the algebras $sl(n)$ model soliton interactions in the adiabatic approximation (Gerdjikov *et al.* , 1996; Uzunov *et al.* , 1996; Gerdjikov *et al.* , 1998).

Below we analyze the algebraic aspects of the Toda chains and describe their families of real Hamiltonan forms. They can be viewed as special cases of the affine Toda field theories widely discussed in the literature (see *e.g.* (Mikhailov, 1981; Mikhailov *et al.* , 1981) and the references therein). It will be clear below that they are related mostly to the normal real form of the Lie algebra $\mathfrak{g}$. Our aim here is to:

1. generalize the TC to complex-valued dynamical variables $\vec{q}^{\mathbb{C}} = \vec{q}_0 + i\vec{q}_1$, and

2. outline the procedure for constructing the RHF of these TC models.

The TC can be written down as finite-dimensional Hamiltonian system:

$$\begin{aligned} &\frac{dq_k}{\mathrm{d}t} = \{q_k, H\}, \qquad \frac{\mathrm{d}p_k}{dt} = \{p_k, H\}, \\ &H_{\mathrm{TC}} = \frac{1}{2}(\vec{p}(t), \vec{p}(t)) + \sum_{k=0}^{r} n_k e^{(\vec{q}(t), \alpha_k)}. \end{aligned} \tag{20}$$

Here α_k, $k = 1, \dots, r$ are the simple roots of the simple Lie algebra $\mathfrak{g}$, α_0 is the minimal root and n_k are the positive integers for which $\alpha_0 = \sum_{k=1}^{r} n_k \alpha_k$, see (Helgasson, 2001). The canonical momenta $\vec{p}$ and coordinates $\vec{q}$ satisfy canonical Poisson brackets:

$$\{q_k(t), p_j(t)\} = \delta_{jk}. \tag{21}$$

Remark 1 *Changing the variables* $q_k \to q_k + \frac{2\omega_k}{(\alpha_k,\alpha_k)} \ln n_k$ *where* ω_k *are the fundamental weights of* $\mathfrak{g}$ *can make all* n_k *in eq. (20) equal to 1.*

Next we impose the involution $\mathcal{C}$ on the phase space $\mathcal{M} \equiv \{q_k, p_k\}_{k=1}^{n}$ satisfying:

$$\begin{aligned} &1) \qquad \mathcal{C}(F(p_k, q_k)) = F(\mathcal{C}(p_k), \mathcal{C}(q_k)), \\ &2) \qquad \mathcal{C}\left(\{F(p_k, q_k), G(p_k, q_k)\}\right) = \{\mathcal{C}(F), \mathcal{C}(G)\}, \\ &3) \qquad \mathcal{C}(H(p_k, q_k)) = H(p_k, q_k). \end{aligned} \tag{22}$$

It is important also that the Hamiltonian $H(p_k, q_k)$ is an analytic function of the dynamical variables $q_k(t)$ and $p_k(t)$. The complexification of the TC is rather straightforward. The resulting complex TC can be written down as standard Hamiltonian system with twice as many dynamical variables $\vec{q}_a(t)$, $\vec{p}_a(t)$, $a = 0, 1$:

$$\vec{p}^{\mathbb{C}}(t) = \vec{p}^{\,0}(t) + i\vec{p}^{\,1}(t), \qquad \vec{q}^{\mathbb{C}}(t) = \vec{q}^{\,0}(t) + i\vec{q}^{\,1}(t), \tag{23}$$

$$\{q_k^0(t), p_j^0(t)\} = -\{q_k^1(t), p_j^1(t)\} = \delta_{kj}. \tag{24}$$

The corresponding Hamiltonian and symplectic form equal

$$
\begin{aligned}
\mathcal{H}^{\mathbb{C}}_{\rm CTC} &\equiv \mathrm{Re}\,\mathcal{H}_{\rm TC}(\vec{p}^{\,0}+i\vec{p}^{\,1},\vec{q}^{\,0}+i\vec{q}^{\,1}) \\
&= \frac{1}{2}(\vec{p}^{\,0},\vec{p}^{\,0}) - \frac{1}{2}(\vec{p}^{\,1},\vec{p}^{\,1}) + \sum_{k=0}^{r} n_k e^{(\vec{q}^{\,0},\alpha_k)}\cos((\vec{q}^{\,1},\alpha_k)), \qquad (25)\\
\omega^{\mathbb{C}} &= (d\vec{p}^{\,0}\wedge i d\vec{q}^{\,0} - d\vec{p}^{\,1}\wedge d\vec{q}^{\,1}).
\end{aligned}
$$

The family of RHF then are obtained from the CTC by imposing an invariance condition with respect to the involution $\tilde{\mathcal{C}} \equiv \mathcal{C}\circ *$ where by $*$ we denote the complex conjugation. The involution $\tilde{\mathcal{C}}$ splits the phase space $\mathcal{M}^{\mathbb{C}}$ into a direct sum $\mathcal{M}^{\mathbb{C}} \equiv \mathcal{M}^{\mathbb{C}}_{+}\oplus\mathcal{M}^{\mathbb{C}}_{-}$ where

$$
\mathcal{M}^{\mathbb{C}}_{+} = \mathcal{M}_0 \oplus i\mathcal{M}_1, \qquad \mathcal{M}^{\mathbb{C}}_{-} = i\mathcal{M}_0 \oplus \mathcal{M}_1, \qquad (26)
$$

The phase space of the RHF is $\mathcal{M}_{\mathbb{R}} \equiv \mathcal{M}^{\mathbb{C}}_{+}$. By $\mathcal{M}_0$ and $\mathcal{M}_1$ we denote the eigensubspaces of $\mathcal{C}$, *i.e.* $\mathcal{C}(u_a) = (-1)^a u_a$ for any $u_a \in \mathcal{M}_a$.

Each involution $\mathcal{C}$ satisfying 1) - 3) can be related to a RHF of the TC. Condition 3) means that $\mathcal{C}$ must preserve the system of admissible roots of $\mathfrak{g}$; so $\mathcal{C}$ must correspond to $\mathbb{Z}_2$-symmetry of the extended Dynkin diagrams of $\mathfrak{g}$, see (Helgasson, 2001; Olive *et al.* , 1986). In (Gerdjikov *et al.* , 2002, 2004) we gave examples of RHF for TC related to the simple Lie algebras. Here, beside the examples of the $sl(5)$ and $so(5)$-TC, we analyze the TC related to the Kac-Moody algebra $A_4^{(2)}$.

4.1. RHF OF TODA CHAINS RELATED TO $A_{2r} \simeq sl(2r+1,\mathbb{C})$ LIE ALGEBRAS

We start with some basic facts about the algebra $\mathfrak{g} \simeq A_{2r} \simeq sl(2r+1)$ (Helgasson, 2001). The set of roots of the algebra is $\Delta \equiv \{\alpha = e_j - e_k\}$, $1 \le j \neq k \le 2r+1$. The set of simple roots is $\pi_{A_{2r}} \equiv \{\alpha_k = e_k - e_{k+1}, k = 1,\dots,2r\}$. The Cartan-Weyl basis in the typical representation of $sl(2r+1)$ is given by $2r+1\times 2r+1$-matrices:

$$
H_k = E_{k,k} - E_{k+1,k+1}, \qquad E_\alpha = E_{j,k} \qquad (27)
$$

where we used the matrices $(E_{jk})_{mn} = \delta_{jm}\delta_{kn}$. It is well known that each positive root $e_k - e_j$, $k<j$ is a linear combination of simple roots with non-negative coefficients: $\alpha \equiv e_k - e_j = \alpha_k + \alpha_2 + \cdots + \alpha_{j-1}$. The sum of the coefficients in this expansion is known as the height of the root: $\mathrm{ht}\,\alpha = j-k$. Next we fix up the involution $\mathcal{C}$ by:

$$
\mathcal{C}(q_k) = -q_{2r+2-k}, \qquad \mathcal{C}(p_k) = -p_{2r+2-k}, \qquad k = 1,\dots,r, \qquad (28)
$$

The coordinates in $\mathcal{M}_\pm$ are given by $p_k = p_k^+ + ip_k^-$, $q_k = q_k^+ + iq_k^-$:

$$
q_k^\pm = \frac{1}{\sqrt{2}}(q_k \mp q_{2r+2-k}), \qquad p_k^\pm = \frac{1}{\sqrt{2}}(p_k \mp p_{2r+2-k}), \qquad (29)
$$

where $k = 1, \dots, r$, *i.e.* , $\dim \mathcal{M}_+ = \dim \mathcal{M}_- = 2r$. Then the densities $\mathcal{H}^{\mathbb{R}}_{\rm TC}$, $\omega^{\mathbb{R}}_{\rm TC}$ for the RHF of TC equal:

$$\begin{aligned}
\mathcal{H}^{\mathbb{R}}_{1,\rm TC} &= \frac{1}{2}\sum_{k=1}^{r}\left(p_k^{+2} - p_k^{-2}\right) + \sum_{k=1}^{r-1} 2e^{(q_k^+ - q_{k+1}^+)/\sqrt{2}} \cos\left(\frac{q_k^- - q_{k+1}^-}{\sqrt{2}}\right) \\
&\quad + e^{q_r^+/\sqrt{2}} \cos\left(\frac{q_{r-1}^- - q_r^-}{\sqrt{2}}\right), \\
\omega^{\mathbb{R}}_{1,\rm TC} &= \sum_{k=1}^{r}\left(dp_k^+ \wedge dq_k^+ - dp_k^- \wedge dq_k^-\right).
\end{aligned} \tag{30}$$

where $\vec{p}_k^{\pm} = \mathrm{d}\vec{q}_k^{\pm}/\mathrm{d}t$. If we put $q_j^- = 0$, $p_j^- = 0$ we get the reduced TC related to the algebra B_r (or to the height 1 Kac-Moody algebra $B_r^{(1)}$ (Olshanetzky *et al.* , 1983).

In what follows, for simplicity we will assume $r = 2$; generalization for $r > 2$ is straightforward. Thus we start with $sl(5)$ whose Coxeter number is 5. Let us introduce in it grading using its Coxeter automorphism:

$$C_1 = \exp\left(\frac{2\pi i}{5} H_{\vec{\rho}}\right), \qquad \vec{\rho} = \sum_{s=1}^{r} \omega_s = 2e_1 + e_2 - e_4 - 2e_5. \tag{31}$$

Grading of $sl(5)$ according to C_1 splits it into 5 linear subspaces

$$sl(5) \equiv \mathop{\oplus}_{s=0}^{4} \mathfrak{g}^{(s)}, \tag{32}$$

where $\mathfrak{g}^{(s)}$ is the eigensubspace of C_1 corresponding to the eigenvalue ω_1^s, $\omega_1 = \exp(2\pi i/5)$. More specifically $\mathfrak{g}^{(k)}$ is the set of roots of height k:

$$\mathfrak{g}^{(0)} \equiv \mathfrak{h}, \qquad \mathfrak{g}^{(k)} \equiv \text{l.c.}\{e_j - e_{j+k}\}, \quad k = 1, \dots, 4 \tag{33}$$

and $j + k$ is taken modulo 5. The Lax pair of the corresponding TC is given by (Manakov, 1974; Flaschka, 1974):

$$\begin{aligned}
L_1 &= \frac{1}{2}\sum_{s=1}^{5} p_k H_{e_k} + \sum_{s=1}^{4} a_s (E_{\alpha_s} + E_{-\alpha_s}) + c_0 a_0 (E_{\alpha_0} + E_{-\alpha_0}), \\
M_1 &= \sum_{s=1}^{4} a_s (E_{\alpha_s} - E_{-\alpha_s}) + c_0 a_0 (E_{\alpha_0} - E_{-\alpha_0}),
\end{aligned} \tag{34}$$

where $a_k = 1/2 \exp((\alpha_k, \vec{q})/2)$. The parameter c_0 above takes values 0 and 1 corresponding to the two types of Toda chains: conformal and affine respectively. The Lax equation $dL_1/dt = [L_1, M_1]$ is equivalent to the set of equations:

$$\frac{da_k}{dt} = (p_{k+1} - p_k) a_k, \qquad \frac{dp_k}{dt} = 2(a_k^2 - a_{k-1}^2), \tag{35}$$

where $k = 1, \ldots, 5$ and $k+1$ is taken modulo 5. These are the TC equations for the $sl(5)$-algebra with Hamiltonian:

$$H_1 \equiv 2\mathrm{tr}\, L^2 := \sum_{k=1}^{5} \frac{p_k^2}{2} + \sum_{k=1}^{4} e^{(\alpha_k, \vec{q})} + c_0 e^{(\alpha_0, \vec{q})}. \tag{36}$$

4.2. REAL HAMILTONIAN FORMS OF $B_2 \simeq so(5)$ TODA CHAIN

Our next example is related to the $so(5)$ Lie algebra. It can be obtained from $sl(5)$ by imposing invariance with respect to the external automorphism V corresponding to the symmetry of the Dynkin diagram of $sl(5)$:

$$V(X) = -S_0 X^T S_0^{-1}, \qquad S_0 = \sum_{s=1}^{5} (-1)^{s+1} E_{s,6-s}, \tag{37}$$

The set of roots of $so(5)$ is $\Delta_2 \equiv \{\pm(e_1 \pm e_2), \pm e_1, \pm e_2\}$. The Cartan-Weyl basis which is compatible with the automorphism V, takes the form:

$$\begin{aligned} &\mathcal{H}_1^+ = E_{11} - E_{55}, \qquad \mathcal{H}_2^+ = E_2 - E_{44}, \\ &\mathcal{E}_{kj}^+ = E_{kj} - S_0 E_{jk} S_0 = E_{kj} - (-1)^{k+j} E_{\bar{j},\bar{k}}, \end{aligned} \tag{38}$$

$\mathcal{E}_{12}^+$ and $\mathcal{E}_{14}^+$ correspond to the roots $e_1 - e_2$ and $e_1 + e_2$ respectively; $\mathcal{E}_{13}^+$ and $\mathcal{E}_{23}^+$ correspond to e_1 and e_2. The Weyl generators for the negative roots are obtained by transposition.

Again we can introduce grading using the Coxeter automorphism. However, now the Coxeter number $h = 4$ and the automorphism is provided by

$$C_2 = \exp\left(\frac{2\pi i}{4} H_{\vec{\rho}}\right), \qquad \vec{\rho} = 2e_1 + e_2, \tag{39}$$

Now the grading is according to the height of the roots of $so(5)$ modulo 4. The minimal (resp. maximal) roots is $-e_1 - e_2$ (resp. $e_1 + e_2$). If $c_0 = 1$ we have

$$\begin{aligned} L_2 = &\frac{1}{2} p_1 \mathcal{H}_1^+ + \frac{1}{2} p_2 \mathcal{H}_2^+ + a_1 (E_{e_1 - e_2} + E_{-e_1 + e_2}) + a_2 (E_{e_2} + E_{-e_2}) \\ &+ a_0 (E_{-e_1 - e_2} + E_{e_1 + e_2}), \\ M_2 = {}& a_1 (E_{e_1 - e_2} - E_{-e_1 + e_2}) + a_2 (E_{e_2} - E_{-e_2}) + c_0 a_0 (E_{-e_1 - e_2} - E_{e_1 + e_2}). \end{aligned} \tag{40}$$

The Hamiltonian of the $so(5)$ TC is

$$H := \mathrm{tr}\, L^2 = \frac{1}{2}\left(p_1^2 + p_2^2\right) + 4(a_1^2 + a_2^2 + a_0^2). \tag{41}$$

where $a_1 = \frac{1}{2}\exp((q_1 - q_2)/2)$, $a_2 = \frac{1}{2}\exp(q_2/2)$ and $a_0 = \frac{1}{2}\exp(-(q_1 + q_2)/2)$.

The set of equations:

$$
\begin{aligned}
&\frac{\mathrm{d}a_1}{\mathrm{d}t} = (p_1 - p_2)a_1, \qquad &&\frac{\mathrm{d}a_2}{\mathrm{d}t} = p_2 a_2, \qquad &&\frac{\mathrm{d}a_0}{\mathrm{d}t} = -(p_1 + p_2)a_0,\\
&\frac{\mathrm{d}p_1}{\mathrm{d}t} = 4(a_3^2 - a_1^2), \qquad &&\frac{\mathrm{d}p_2}{\mathrm{d}t} = 4(a_1^2 - a_2^2 + a_0^2),
\end{aligned} \tag{42}
$$

If we put $p_k = \mathrm{d}q_k/\mathrm{d}t$ then we get the affine Toda chain related to $so(5)$:

$$
\begin{aligned}
\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} &= -\left(e^{q_1 - q_2} - e^{-(q_1+q_2)}\right),\\
\frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} &= e^{(q_1 - q_2)} - e^{q_2} + e^{-(q_1+q_2)}.
\end{aligned} \tag{43}
$$

The RHF of this Toda chain can be generated using the $\mathbb{Z}_2$-automorphism V_2 of the extended Dynkin diagram which acts as follows: $V_2 e_1 = -e_1$, $V_2 e_2 = e_2$. The Hamiltonian for the RHF of this TC is given by (compare with (41)):

$$
H_{\mathrm{RHF}} = \frac{1}{2}(-p_1^2 + p_2^2) + e^{q_2} + 2e^{-q_2}\cos(q_1). \tag{44}
$$

4.3. REAL HAMILTONIAN FORMS OF $A_4^{(2)}$ TODA CHAIN

The last example corresponds to the grading of the height 2 (twisted) Kac-Moody algebra $A_4^{(2)}$. The grading in $\mathfrak{g} \simeq sl(5)$ is introduced by $C_3 \equiv C_1 \cdot V$, where C_1 is the Coxeter automorphism of $sl(5)$ (31) and V is induced by the symmetry of the Dynkin diagram. This means that $Ve_k = -e_{6-k}$. Note that V preserves the subspaces $\mathfrak{g}^{(s)}$ in (32). As a result each of these subspaces is split into $\mathfrak{g}^{(s)} = \mathfrak{g}_+^{(s)} \oplus \mathfrak{g}_-^{(s)}$ so that $VX_\pm^{(s)} = \pm X_\pm^{(s)}$ for all $X_\pm^{(s)} \in \mathfrak{g}_\pm^{(s)}$. Obviously $sl(5)$ is split now into 10 linear subspaces of C_3 and $C_3^{10} = \mathbb{1}$. So the eigenvalues of C_3 are ω_0^p where $\omega = \exp(2\pi i/10)$. In what follows we will need the basis in $\mathfrak{g}_+^{(0)}$, $\mathfrak{g}_-^{(1)}$ and $\mathfrak{g}_-^{(-1)}$.

$$
\mathfrak{g}_+^{(0)} \simeq \text{l. c.}\{\mathcal{H}_1^+, \mathcal{H}_2^+\} \tag{45}
$$

Let us now introduce basis which is compatible with the automorphism V, namely:

$$
\begin{aligned}
\mathcal{E}_{kj}^+ &= E_{kj} - S_0 E_{jk} S_0 = E_{kj} - (-1)^{k+j} E_{\bar{j},\bar{k}}, \qquad && V(\mathcal{E}_{kj}^+) = \mathcal{E}_{kj}^+,\\
\mathcal{E}_{kj}^- &= E_{kj} + S_0 E_{jk} S_0 = E_{kj} + (-1)^{k+j} E_{\bar{j},\bar{k}}, \qquad && V(\mathcal{E}_{kj}^-) = -\mathcal{E}_{kj}^-.
\end{aligned} \tag{46}
$$

Obviously, using V each of the subspaces splits into

$$
\mathfrak{g}^{(k)} = \mathfrak{g}_+^{(k)} \oplus \mathfrak{g}_-^{(k)}, \quad \mathfrak{g}_-^{(3)} = \text{l. c.}\{\mathcal{E}_{14}^-, \mathcal{E}_{31}^-, \mathcal{E}_{42}^-\}, \quad \mathfrak{g}_-^{(2)} = \text{l. c.}\{\mathcal{E}_{13}^-, \mathcal{E}_{41}^-, \mathcal{E}_{24}^-\}, \tag{47}
$$

Since $C_3(\mathfrak{g}_+^{(0)}) \equiv \mathfrak{g}_+^{(0)}$, $C_3(\mathfrak{g}_-^{(3)}) \equiv \omega_0 \mathfrak{g}_-^{(3)}$ and $C_3(\mathfrak{g}_-^{(2)}) \equiv \omega_0^{-1} \mathfrak{g}_-^{(2)}$, the Lax pair is:

$$
\begin{aligned}
L &= b_1 \mathcal{H}_1 - b_2 \mathcal{H}_2 + a_1(\mathcal{E}_{14}^- + \mathcal{E}_{41}^-) + a_2(\mathcal{E}_{13}^- + \mathcal{E}_{31}^-) + a_0(\mathcal{E}_{24}^- + \mathcal{E}_{42}^-),\\
M &= a_1(\mathcal{E}_{14}^- - \mathcal{E}_{41}^-) + a_2(\mathcal{E}_{13}^- - \mathcal{E}_{31}^-) + a_0(-\mathcal{E}_{24}^- + \mathcal{E}_{42}^-).
\end{aligned} \tag{48}
$$

The set of equations:

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \frac{1}{2}(p_1 - p_2)a_1, \qquad \frac{\mathrm{d}a_2}{\mathrm{d}t} = p_2 a_2, \qquad \frac{\mathrm{d}a_0}{\mathrm{d}t} = -\frac{1}{2}p_2 a_2,$$
$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = 4(a_0^2 - a_1^2), \qquad \frac{\mathrm{d}p_2}{\mathrm{d}t} = 4(a_1^2 - 2a_2^2), \tag{49}$$

If we put

$$p_k = \frac{\mathrm{d}q_k}{\mathrm{d}t}, \qquad a_1 = \frac{1}{2}e^{(q_1-q_2)/2}, \qquad a_2 = \frac{1}{2}e^{-q_1/2}, \qquad a_0 = \frac{1}{2}e^{q_1}, \tag{50}$$

then we get the affine Toda chain related to the twisted Kac-Moody algebra $A_4^{(2)}$:

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}t^2} = -e^{q_1-q_2} + e^{-q_1}, \qquad \frac{\mathrm{d}^2 q_2}{\mathrm{d}t^2} = e^{q_1-q_2} - 4e^{2q_2}. \tag{51}$$

The Hamiltonian is equal to tr L^2 and is given by:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + e^{q_1-q_2} + e^{-q_1} + 2e^{2q_2}. \tag{52}$$

5. CONCLUSIONS

The examples of Toda chains can be generalized to include also the other twisted Kac-Moody algebras. The RHF of these TC are generated by automorphisms, which are symmetries of the extended Dynkin diagrams. For the algebras B_r such symmetry exists only for $r = 2$. Another problem is to study the RHF of other classes of dynamical systems, such as Hénon-Heiles-type systems (Kostov *et al.* , 2010), the complex symmetric Hamiltonian systems proposed in (Ivanov, 2006) etc.

Acknowledgements. VG and RI acknowledge financial support from the Irish Research Council (Ireland). VG thanks A. Stefanov and S. Varbev for useful discussions. GV acknowledges financial support from the Agenzia Spaziale Italiana (ASI).

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Received on 20 June 2014