NEW REGULARIZATION OF THE RESTRICTED THREE-BODY PROBLEM

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Abstract. We find a very interesting transformation for regularization of the coordinates and time for the restricted three-body problem. The new regularization can give us significant informations regarding the behavior of the dynamical system near the singularity point.

Key words: Celestial Mechanics; Regularization.

1. INTRODUCTION

Regularization is originally defined as the elimination of singularities occurring in the equations of motion by properly selected variables (Szebehely, 1967). From Newton's law, we know that the bodies interact by means of a force which is proportional to the inverse of the squared distance. As the two bodies approach each other (close approach), their distance tends to zero and, consequently, the differential equation describing the dynamics of the system becomes singular when the two bodies collide. From the theoretical point of view, the singularity due to binary collisions between point masses can be handled by means of the regularization theory (Waldvogel, 1972; Érdi, 2004).

We know a lot of methods in the regularization theory for the 2-body problem, for instance the Euler method (Euler, 1765), Levi-Civita method (Levi-Civita, 1906) and the Kustaanheimo-Stiefel method (Kustaanheimo *et al.*, 1965). Many studies of the regularization problem are in the restricted 3-body problem, where we have 2 singularities. We can regularize local (one of them), or global. Birkhoff (1915); Thiele (1896); Burrau (1906); Lemaître (1955); Arenstorf (1963) and many other researchers studied the regularization of the restricted three-body problem (Aarseth *et al.*, 1974).

In this article we construct a new regularizing transformation, which depends on the right selection of the coordinates and time transformation. This new regularizing transformation has its own advantage compared to the Levi-Civita regularization, because it preserves the form of the regularized trajectory.

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2. THE RESTRICTED THREE-BODY PROBLEM

Denoting S_1 and S_2 the components of the binary system (whose masses are m_1 and m_2), the equations of motion of the test particle (in the frame of the restricted three-body problem) in the coordinate system (S_1, x, y, z), (the physical plane) are (Roman *et al.*, 2012; Roman, 2011):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} = x - \frac{q}{1+q} - \frac{x}{(1+q)r_1^3} - \frac{q(x-1)}{(1+q)r_2^3} \tag{1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} = y - \frac{y}{(1+q)r_1^3} - \frac{q\,y}{(1+q)r_2^3} \tag{2}$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = -\frac{z}{(1+q)r_1^3} - \frac{q\,z}{(1+q)r_2^3} \tag{3}$$

where

$$r_1 = \sqrt{x^2+y^2+z^2}\ , \quad r_2 = \sqrt{(x-1)^2+y^2+z^2}\ , \quad q = \frac{m_2}{m_1}. \tag{4}$$

These equations have singularities in the terms $\frac{1}{r_1}$ and $\frac{1}{r_2}$ (Mioc *et al.*, 2002; Csillik, 2003; Waldvogel, 1982, 2006). In order to regularize the equations (1)-(3), we introduce the generalized coordinates q_1, q_2, q_3 and generalized momenta p_1, p_2, p_3 (Boccaletti *et al.*, 1996; Roman *et al.*, 2012), and write the Hamiltonian and the canonical equations of motion:

$$\mathcal{H} = \frac{1}{2}(p_1^2+p_2^2+p_3^2) + p_1q_2 - q_1p_2 + \frac{q_1^2}{2} + \frac{q_2^2}{2} - \psi(q_1,q_2,q_3)\ . \tag{5}$$

Here

$$\Psi(q_1,q_2,q_3) = \frac{1}{2}\left[\left(q_1 - \frac{q}{1+q}\right)^2 + q_2^2 + \frac{2}{(1+q)r_1} + \frac{2q}{(1+q)r_2}\right]\ , \tag{6}$$

with

$$r_1 = \sqrt{q_1^2+q_2^2+q_3^2}, \quad r_2 = \sqrt{(q_1-1)^2+q_2^2+q_3^2}\ . \tag{7}$$

The generalized coordinates and the generalized momenta were:

$$q_1 = x\ ,\ q_2 = y\ ,\ q_3 = z\ ,\ p_1 = \dot{q_1} - q_2 \ \ p_2 = \dot{q_2} + q_1\ ,\ p_3 = \dot{q_3}. \tag{8}$$

The canonical equations have the general form:

$$\dot{q_i} = \frac{\partial\mathcal{H}}{\partial p_i}\ , \ \ \dot{p_i} = -\frac{\partial\mathcal{H}}{\partial q_i}\ , \quad i \in \{1,2,3\} \tag{9}$$

The canonical equations obtained from equations (1)-(3) have, in the (S_1, q_1, q_2, q_3) coordinate system, the explicit form:

$$\frac{\mathrm{d}q_1}{\mathrm{d}t} = p_1 + q_2 \tag{10}$$

$$\frac{\mathrm{d}q_2}{\mathrm{d}t} = p_2 - q_1 \tag{11}$$

$$\frac{\mathrm{d}q_3}{\mathrm{d}t} = p_3 \tag{12}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}t} = p_2 - \frac{q}{1+q} - \frac{1}{1+q} \cdot \frac{q_1}{r_1^3} - \frac{q}{1+q} \cdot \frac{q_1 - 1}{r_2^3} \tag{13}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}t} = -p_1 - \frac{1}{1+q} \cdot \frac{q_2}{r_1^3} - \frac{q}{1+q} \cdot \frac{q_2}{r_2^3} \tag{14}$$

$$\frac{\mathrm{d}p_3}{\mathrm{d}t} = -\frac{1}{1+q} \cdot \frac{q_3}{r_1^3} - \frac{q}{1+q} \cdot \frac{q_3}{r_2^3}. \tag{15}$$

For simplicity, we shall consider in what follows that the third body moves into the orbital plane ($z = 0$).

3. THE LEVI-CIVITA REGULARIZATION

We briefly present the well-known Levi-Civita regularization methods (Roman *et al.*, 2012). For the regularization of the equations of motion in the (q_1, S_1, q_2) coordinate system, we shall introduce new variables Q_1 and Q_2, connected with the coordinates q_1 and q_2 by the Levi-Civita equations (Levi-Civita, 1906):

$$q_1 = Q_1^2 - Q_2^2\,, \quad q_2 = 2Q_1Q_2\,, \tag{16}$$

Using Levi-Civita's coordinate transformation $f = q_1 = Q_1^2 - Q_2^2\,,\ \ g = q_2 = 2Q_1Q_2$, the equations of motion of the restricted three-body problem becomes:

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{P_1}{D} + \frac{Q_2}{2} \tag{17}$$

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{P_2}{D} - \frac{Q_1}{2} \tag{18}$$

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = \frac{P_2}{2} - \frac{2qQ_1}{1+q} - \frac{2}{1+q}\frac{Q_1}{\bar{r}_1^2} - \frac{2q}{1+q}\frac{Q_1(\bar{r}_1 - 1)}{\bar{r}_2^3} + \frac{(P_1^2 + P_2^2)Q_1}{4\bar{r}_1^2} \tag{19}$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = -\frac{P_1}{2} + \frac{2qQ_2}{1+q} - \frac{2}{1+q}\frac{Q_2}{\bar{r}_1^2} - \frac{2q}{1+q}\frac{Q_2(\bar{r}_1 + 1)}{\bar{r}_2^3} + \frac{(P_1^2 + P_2^2)Q_2}{4\bar{r}_1^2} \tag{20}$$

where

$$\begin{aligned}\bar{r}_1 &= Q_1^2+Q_2^2\\ \bar{r}_2 &= \sqrt{(Q_1^2-Q_2^2-1)^2+4Q_1^2Q_2^2}\end{aligned}$$

with the new Hamiltonian

$$\begin{aligned}\mathcal{H}_{S1} &= \frac{P_1^2+P_2^2}{8(Q_1^2+Q_2^2)}+\frac{1}{2}(P_1Q_2-P_2Q_1)+\frac{q}{1+q}(Q_1^2-Q_2^2)-\frac{1}{1+q}\cdot\\ &\cdot\frac{1}{Q_1^2+Q_2^2}-\frac{q}{1+q}\cdot\frac{1}{\sqrt{(Q_1^2-Q_2^2-1)^2+4Q_1^2Q_2^2}}-\frac{q^2}{2(1+q)^2}\end{aligned} \quad (21)$$

Introducing a time transformation for the new equations of motion (17)-(20), the motion of the system is slowed down, in order to observe and study the movement of the system around the singularity points (Roman *et al.*, 2012; Mikkola *et al.*, 1996; Castilho *et al.*, 1999).

4. THE NEW REGULARIZATION

We propose a new regularizing transformation, to avoid the singularities from the equations of motion (10)–(11), (13)–(14). As in the Levi-Civita regularization method, we have two transformations for the regularization procedure: *coordinate transformation*, which gives the shape of the orbit, and *time transformation*, which makes the slow-down motion (Celletti *et al.*, 2011; Jiménez-Perez *et al.*, 2011).

4.1. COORDINATE TRANSFORMATION

The first step performed in the process of regularization consists in introduction of new coordinates Q_1 and Q_2. Let us introduce the generating function $\mathcal{S}$, (Stiefel *et al.*, 1971):

$$\mathcal{S} = -p_1 f(Q_1,Q_2) - p_2 g(Q_1,Q_2) \quad (22)$$

a $\mathcal{C}^2$ function. Here f and g are harmonic conjugated functions, with the property

$$\begin{aligned}\frac{\partial f}{\partial Q_1} &= \frac{\partial g}{\partial Q_2}\\ \frac{\partial f}{\partial Q_2} &= -\frac{\partial g}{\partial Q_1}\end{aligned}$$

The generating equations are

$$q_i = -\frac{\partial \mathcal{S}}{\partial p_i}, \quad P_i = -\frac{\partial \mathcal{S}}{\partial Q_i}, \quad i \in \{1,2\}, \quad (23)$$

with $P_1,\ P_2$ as new generalized momenta, or explicitly

$$\begin{aligned}
q_1 &= \frac{\partial\mathcal{S}}{\partial p_1} = f(Q_1,Q_2)\\
q_2 &= \frac{\partial\mathcal{S}}{\partial p_2} = g(Q_1,Q_2)\\
P_1 &= \frac{\partial\mathcal{S}}{\partial Q_1} = p_1\frac{\partial f}{\partial Q_1}+p_2\frac{\partial g}{\partial Q_1} = p_1a_{11}+p_2a_{12}\\
P_2 &= \frac{\partial\mathcal{S}}{\partial Q_2} = p_1\frac{\partial f}{\partial Q_2}+p_2\frac{\partial g}{\partial Q_2} = -p_1a_{12}+p_2a_{11}
\end{aligned}\tag{24}$$

where

$$\begin{aligned}
a_{11} &= \frac{\partial f}{\partial Q_1} = \frac{\partial g}{\partial Q_2}\\
a_{12} &= -\frac{\partial f}{\partial Q_2} = \frac{\partial g}{\partial Q_1}
\end{aligned}$$

Let us introduce the following notation:

$$A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},\ \ D = detA = a_{11}^2+a_{22}^2$$

The new Hamiltonian with the generalized coordinates Q_1 and Q_2 and the generalized momenta P_1 and P_2 is:

$$\begin{aligned}
\mathcal{H}(Q_1,Q_2,P_1,P_2) &= \frac{1}{2D}\left[P_1^2+P_2^2+P_1\frac{\partial}{\partial Q_2}(f^2+g^2)-P_2\frac{\partial}{\partial Q_1}(f^2+g^2)\right]+\\
&+ \frac{q}{1+q}f-\frac{1}{1+q}\cdot\frac{1}{\overline{r}_1}-\frac{q}{1+q}\cdot\frac{1}{\overline{r}_2}-\frac{q^2}{2(1+q)^2}
\end{aligned}\tag{25}$$

where

$$\overline{r}_1=\sqrt{f^2+g^2}\ ,\ \ \overline{r}_2=\sqrt{(f-1)^2+g^2}\ ,\ \ D=\left(\frac{\partial f}{\partial Q_1}\right)^2+\left(\frac{\partial g}{\partial Q_1}\right)^2.$$

In our new method of regularization, the new variables Q_1 and Q_2, are connected with the coordinates q_1 and q_2 by the *hyperbolical transformation*:

$$q_1=\sin(Q_1)\cosh(Q_2)\ ,\quad q_2=\cos(Q_1)\sinh(Q_2).\tag{26}$$

The equations (26) transform the points $S_1(0;0)$ and $S_2(1;0)$ from the physical plane, into the points $S_1(0;0)$ and $S_2(1.57;0)$ in the regularized plane.

Then, the canonical equations (10)–(11), (13)–(14) become:

$$\frac{\mathrm{d}Q_1}{\mathrm{d}t} = \frac{P_1}{D} + \frac{\sinh(2Q_2)}{2D} \tag{27}$$

$$\frac{\mathrm{d}Q_2}{\mathrm{d}t} = \frac{P_2}{D} - \frac{\sin(2Q_1)}{2D} \tag{28}$$

$$\begin{aligned}\frac{\mathrm{d}P_1}{\mathrm{d}t} &= -\frac{P_1^2+P_2^2}{2}\cdot\frac{\sin(2Q_1)}{D^2} - \frac{P_1}{2}\cdot\frac{\sinh(2Q_2)\sin(2Q_1)}{D^2} + \\ &+ \frac{P_2}{D}\cdot\cos(2Q_1) + \frac{P_2}{2}\cdot\frac{\sin^2(2Q_1)}{D^2} - \\ &- \frac{q}{1+q}\cdot\cos(Q_1)\cosh(Q_2) - \frac{1}{2(1+q)}\cdot\frac{\sin(2Q_1)}{\overline{r}_1^3} + \\ &+ \frac{q}{1+q}\cdot\frac{\cos(Q_1)\cdot(\sin(Q_1)-\cosh(Q_2))}{\overline{r}_2^3}\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\mathrm{d}P_2}{\mathrm{d}t} &= \frac{P_1^2+P_2^2}{2}\cdot\frac{\sinh(2Q_2)}{D^2} + \frac{P_1}{2}\cdot\frac{\sinh^2(2Q_2)}{D^2} - \frac{P_1}{D}\cdot\cosh(2Q_2) - \\ &- \frac{P_2}{2}\cdot\frac{\sin(2Q_1)\sinh(2Q_2)}{D^2} - \frac{q}{1+q}\cdot\sin(Q_1)\sinh(Q_2) - \\ &- \frac{1}{2(1+q)}\cdot\frac{\sinh(2Q_2)}{\overline{r}_1^3} + \frac{q}{1+q}\cdot\frac{\sinh(Q_2)\cdot(\sin(Q_1)-\cosh(Q_2))}{\overline{r}_2^3}\end{aligned} \tag{30}$$

where

$$\overline{r}_1 = \sqrt{\frac{\cosh(2Q_2)-\cos(2Q_1)}{2}}\,,\ \overline{r}_2 = |\cosh(Q_2)-\sin(Q_1)|\,,\ D = \cos^2(Q_1)+\sinh^2(Q_2)$$

with the new Hamiltonian:

$$\begin{aligned}\mathcal{H}_{S1} &= \frac{1}{2D}\left[P_1^2+P_2^2+P_1\sinh(2Q_2)-P_2\sin(2Q_1)\right] + \\ &+ \frac{q}{1+q}\sin(Q_1)\cosh(Q_2) - \frac{1}{1+q}\cdot\frac{1}{\overline{r}_1} - \frac{q}{1+q}\cdot\frac{1}{\overline{r}_2} - \frac{q^2}{2(1+q)^2}\end{aligned} \tag{31}$$

For the hyperbolic regularization we can postulate the following theorem:

Theorem

The hyperbolic regularization preserves the form of the trajectory of the test particle, if the initial coordinates Q_{10} and Q_{20} have absolute values smaller than 1.

Proof:

Expanding into Taylor's series the equations (26) writen for the initial position, we obtain:

$$q_{10} = Q_{10} + \frac{Q_{10}\cdot Q_{20}^2}{2} - \frac{Q_{10}^3}{6} + \frac{Q_{10}\cdot Q_{20}^4}{24} - \frac{Q_{10}^3\cdot Q_{20}^2}{12} + \frac{Q_{10}^5}{120} + \frac{Q_{10}\cdot Q_{20}^6}{720} - \frac{Q_{10}^7}{5040}\dots$$

$$q_{20} = Q_{20} + \frac{Q_{20}\cdot Q_{10}^2}{2} - \frac{Q_{20}^3}{6} + \frac{Q_{20}\cdot Q_{10}^4}{24} - \frac{Q_{20}^3\cdot Q_{10}^2}{12} + \frac{Q_{20}^5}{120} + \frac{Q_{20}\cdot Q_{10}^6}{720} - \frac{Q_{20}^7}{5040}\dots$$

If $Q_{10} < 1$ and $Q_{20} < 1$ we obtain $q_{10} \approx Q_{10}$ and $q_{20} \approx Q_{20}$. So, the initial positions into the physical plane and into the hyperbolical regularized plane have similar coordinates. Idem for initial momenta. So, the trajectories are resemblant.

4.2. TIME TRANSFORMATION

The second step performed in the process of regularization consists in the time transformation. In order to solve the Hamiltonian equations (27)-(30), we introduce the fictitious time τ, and making the time transformation $\frac{dt}{d\tau} = \overline{r}_1^3 \overline{r}_2^2$, the new regular equations of motion become:

$$
\begin{aligned}
\frac{\mathrm{d}Q_1}{\mathrm{d}\tau} &= \left(\frac{P_1}{D} + \frac{\sinh(2Q_2)}{2D}\right)\overline{r}_1^3\overline{r}_2^2 \\
\frac{\mathrm{d}Q_2}{\mathrm{d}\tau} &= \left(\frac{P_2}{D} - \frac{\sin(2Q_1)}{2D}\right)\overline{r}_1^3\overline{r}_2^2 \\
\frac{\mathrm{d}P_1}{\mathrm{d}t} &= \left(-\frac{P_1^2+P_2^2}{2}\cdot\frac{\sin(2Q_1)}{D^2} - \frac{P_1}{2}\cdot\frac{\sinh(2Q_2)\sin(2Q_1)}{D^2} + \frac{P_2}{D}\cdot\cos(2Q_1) + \right. \\
&+ \frac{P_2}{2}\cdot\frac{\sin^2(2Q_1)}{D^2} - \frac{q}{1+q}\cdot\cos(Q_1)\cosh(Q_2) - \frac{1}{2(1+q)}\cdot\frac{\sin(2Q_1)}{\overline{r}_1^3} + \\
&+ \left.\frac{q}{1+q}\cdot\frac{\cos(Q_1)\cdot(\sin(Q_1)-\cosh(Q_2))}{\overline{r}_2^3}\right)\overline{r}_1^3\overline{r}_2^2 \\
\frac{\mathrm{d}P_2}{\mathrm{d}t} &= \left(\frac{P_1^2+P_2^2}{2}\cdot\frac{\sinh(2Q_2)}{D^2} + \frac{P_1}{2}\cdot\frac{\sinh^2(2Q_2)}{D^2} - \frac{P_1}{D}\cdot\cosh(2Q_2) - \right. \\
&- \frac{P_2}{2}\cdot\frac{\sin(2Q_1)\sinh(2Q_2)}{D^2} - \frac{q}{1+q}\cdot\sin(Q_1)\sinh(Q_2) - \frac{1}{2(1+q)}\cdot \\
&\cdot \left.\frac{\sinh(2Q_2)}{\overline{r}_1^3} + \frac{q}{1+q}\cdot\frac{\sinh(Q_2)\cdot(\sin(Q_1)-\cosh(Q_2))}{\overline{r}_2^3}\right)\overline{r}_1^3\overline{r}_2^2 \qquad (32)
\end{aligned}
$$

Now, the equations of motion of the test particle do not have singularities.

5. APPLICATION

In order to emphasize the advantage of the hyperbolical regularization, we made a numerical application, for the Earth-Moon binary system, which is characterized by the mass ratio $q = 0.0123$. We integrated the equations of motion (10)–(11) and (13)–(14) in the physical plane (S_1, q_1, q_2), considering the initial conditions:

$$q_{10} = -0.5\ , \quad q_{20} = 0.1\ , \quad p_{10} = -0.5\ , \quad p_{20} = -0.5\ .$$

The trajectory of a mass point is given in Figure 1, where P_0 represents the initial position. The initial conditions for equations (27)-(30) are obtained by solving the systems:

$$\begin{cases} q_{10} & = \sin(Q_{10}) \cdot \cosh(Q_{20}) \\ q_{20} & = \cos(Q_{10}) \cdot \sinh(Q_{20}) \end{cases} ,$$

$$\begin{cases} P_{10} & = p_{10}\cos(Q_{10})\cosh(Q_{20}) - p_{20}\sin(Q_{10})\sinh(Q_{20}) \\ P_{20} & = p_{10}\sin(Q_{10})\sinh(Q_{20}) + p_{20}\cos(Q_{10})\cosh(Q_{20}) \end{cases} .$$

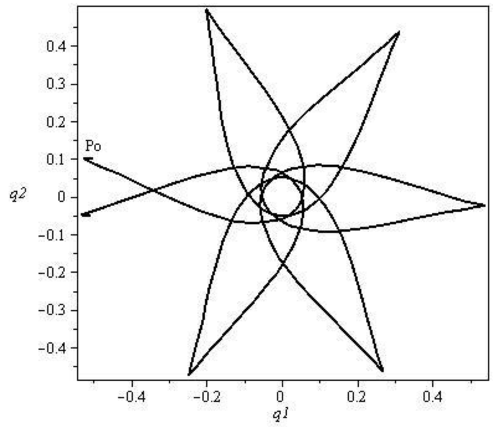

Fig. 1 – The trajectory of the test particle in the physical plane. Here, $P_0(-0.5; 0.1)$.

By consequence, the equations of motion (27)-(30) must be integrated with initial conditions:

$$Q_{10} = -0.52\ , \quad Q_{20} = 0.11\ , \quad P_{10} = -0.46\ , \quad P_{20} = -0.41\ .$$

The trajectory of a mass point in the regularized plane (S_1, Q_1, Q_2) is given in Fig.2a. This trajectory can be compared with the one obtained by applying the Levi-Civita's method, integrating the equations (17)-(20), and is given in Fig.2b.

The initial conditions for equations (17)–(20), (Levi-Civita case) are obtained by solving the systems:

$$\begin{cases} q_{10} & = Q_{10}^2 - Q_{20}^2 \\ q_{20} & = 2Q_{10}Q_{20} \end{cases} , \quad \begin{cases} P_{10} & = 2p_{10}Q_{10} + 2p_{20}Q_{20} \\ P_{20} & = -2p_{10}Q_{20} + 2p_{20}Q_{10} \end{cases} .$$

The corresponding initial conditions in the Levi-Civita case are:

$$Q_{10} = -0.07\ , \quad Q_{20} = -0.71\ , \quad P_{10} = 0.78\ , \quad P_{20} = -0.64\ .$$

It is easy to remark that the hyperbolic regularization preserves the form of the trajectory better than the Levi-Civita method (Fig.1 and 2a are very similar).

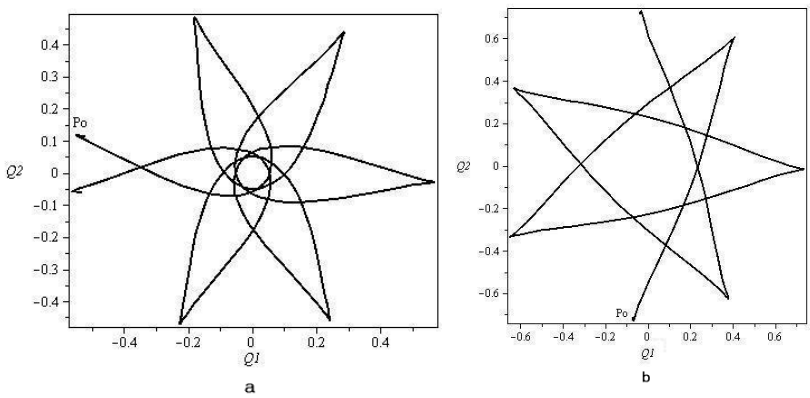

Fig. 2 – The trajectory of the test particle in the regularized plane, using (a) the hyperbolic regularization (where $P_0(-0.52; 0.11)$) and (b) the Levi-Civita regularization (where $P_0(-0.07; -0.71)$).

6. CONCLUSION

Comparing the hyperbolic regularizing transformation and the Levi-Civita regularization, we remark that the shape of the orbit is different. When we want to apply a regularization method, we choose the best method for our problem, the method which gives us more information about the motion near the singularity points. In a critical singularity point, the study in a regularized plane is recommended, because many numerical integrators avoid the singularity point. Our proposed new regularization method, called *the hyperbolical regularization* is a very simple and fast method, which preserves very well the shape of the orbits, and can be effectively used in the phase space to examine a close approach (collision).

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