ON THE VALUE OF THE DYNAMICAL FLATTENING OF THE EARTH

IERONIM MIHAILA[1], ALINA DANIELA VÎLCU[2]

[1] *University of Bucharest, Faculty of Mathematics and Computer Science*
Str. Academiei 14, 010014 Bucharest, Romania
Email: mihaila@fmi.unibuc.ro
[2] *Petroleum-Gas University of Ploiesti*
Department of Information Technology, Mathematics and Physics
Bulevardul Bucuresti 39, 100680 Ploiesti, Romania
Email: daniela.vilcu@upg-ploiesti.ro

Abstract. Firstly it is shown that in the approximation of the homogeneous Earth the dynamical flattening has the greatest value. Next, its value in the case of the elliptical motion of the Moon is deduced, obtaining the value $H = 0.00328022$. Afterwards, it is shown that in the case of Woolard's theory of precession and nutation the value is smaller, namely $H = 0.00327399$. Once with the adoption of the new epoch J2000 and of new values for the constant of general precession, more accurate values for H were obtained. Our analysis revealed that the mean sidereal motion of the Sun must be corrected by the perturbation of the mean longitude at epoch. The formula $p = 1539712.5''H$ per Julian century (p being the constant of lunisolar precession), which allows a good determination of the dynamical flattening, was established. The most probable value of the dynamical flattening is $H = 0.003273795 \pm 1\times10^{-9}$, value adopted by IAU. The secular variation of the flattening is also presented.

Key words: celestial mechanics - Earth's rotation - dynamical flattening - precession.

1. INTRODUCTION

In the last decade of the nineteenth century, Hill (1893) published an important paper on the connection of precession and nutation with the dynamical flattening of the Earth. In this work, instead of the elliptical motion of the Moon, the motion from the theory of Delaunay is adopted.In the same period, Newcomb (1895) published the fundamental work regarding the elements of the four inner planets and the astronomical constants. Using Oppolzer's developments (Oppolzer, 1882), obtained in the case of Hansen's theory of the Moon, he derived for the constant of lunisolar precession and the constant of nutation for the epoch 1850 the following expressions

$$p = [3.68762]\frac{C-A}{C} + [5.937585]\frac{\mu}{1+\mu}\frac{C-A}{C}, \tag{1}$$

$$N = [5.36542]\frac{\mu}{1+\mu}\frac{C-A}{C}, \tag{2}$$

in brackets being the common logarithms of the numerical factors. In the previous formulae, μ is the ratio of the mass of the Moon to that of the Earth, and $(C-A)/C$ is the dynamical flattening. The unit of time is the Julian century. We mention that, in the expressions obtained by Newcomb, the coefficients practically coincide with the ones obtained by Hill, which shows that Delaunay's theory and Hansen's theory describe the motion of the Moon equally accurately. Taking $p = 50.26''$ per Julian year and $N = 9.20''$, Newcomb obtained $(C-A)/C = 0.0032753$. This value was included in the first system of astronomical constants, adopted in 1896, and was replaced with the zonal harmonic coefficient $J_2 = 0.0010827$ in the IAU (1964) System and with the improved value $J_2 = 0.00108263$ in the IAU (1976) System. In the new IAU (2009) System of astronomical constants the last value was replaced with more precise value $J_2 = 0.0010826359$.

The analysis of the perturbations produced in the motion of the artificial satellites led to a more precise knowledge of the Earth's shape and size, as well as its gravitational field. The knowledge of the five coefficients of the second order harmonics of the series of the geopotential allows the determination of the six moments of inertia of the Earth, if another relation between the moments is known. Erzhanov and Kalybaev (1975, 1984) proposed the relation $H = [2C-(A+B)]/2C$, where the dynamical flattening H is known. Therefore, the knowledge of the dynamical flattening is required not only for the more accurate calculation of the nutation series coefficients, but also to determine the moments of inertia of the Earth.

In this paper, using improved values of the astronomical data, we shall determine the dynamical flattening in the case of the elliptical motion of the Moon and in the case of the motion from Brown's theory. Moreover, the values of the flattening obtained by using the more precise theories of the movement of the Sun and Moon and the new values for the precession will be presented and improved. But firstly, we shall calculate the flattening for the homogeneous Earth. Although in the IAU (1976) System of astronomical constants and also in the new IAU (2009) System was adopted the epoch J2000, in the first part of the paper the epoch 1900, January 0, is considered, epoch used in Brown's theory and in Woolard's work (Woolard, 1953), the latter being considered a reference work. On the other hand, the unit of time used in the present paper is Julian century, denoted cy, and the precessional angle is measured in the retrograde sense.

2. THE DYNAMICAL FLATTENING IN THE HOMOGENEOUS EARTH APPROXIMATION

Let $Oxyz$ be the coordinate system defined by the Earth's principal axes of inertia and its center of mass. In this system, the principal moments of inertia have the expressions

$$A = \int_V \delta(x,y,z)(y^2+z^2)\mathrm{d}v, \tag{3}$$

$$B = \int_V \delta(x,y,z)(z^2+x^2)\mathrm{d}v, \tag{4}$$

$$C = \int_V \delta(x,y,z)(x^2+y^2)\mathrm{d}v, \tag{5}$$

where V is the domain occupied by Earth and δ is the density. If the function δ is known, the formulae allow the calculation of the moments and of the dynamical flattening $H = [2C-(A+B)]/2C$, which occurs in Poisson's equations for precession and nutation. The difficulty consists in the fact that δ is not known and it is necessary to make different assumptions about the Earth's internal constitution.

A first approximation is obtained considering that the Earth is homogeneous ($\delta = \mathrm{const.}$). In this case, we have

$$A = \frac{E}{5}(b^2+c^2),\ B = \frac{E}{5}(c^2+a^2),\ C = \frac{E}{5}(a^2+b^2), \tag{6}$$

where E is the mass of the Earth and a, b, c are the semi-axes of the terrestrial ellipsoid. In the case of the ellipsoid of rotation, it follows

$$A = \frac{E}{5}(a^2+c^2),\ C = \frac{2Ea^2}{5}, \tag{7}$$

and one obtains

$$H = \frac{C-A}{C} = \frac{a^2-c^2}{2a^2}, \tag{8}$$

a being the equatorial radius, and c the polar radius. Introducing the geometrical flattening $f = (a-c)/a$, one obtains $H = f - f^2/2$. Note that the dynamical flattening is less than the geometrical one. Using the value $f = 1/298.257 = 0.00335231$ from the IAU (1976) System, it follows $H = 1/299.260 = 0.00334157$. If the new value $f = 1/298.256$ from IAU (2009) System is used, it results $H = 0.00334720$. An improved value of H can be obtained considering the Earth consisting of homogeneous layers and δ a piecewise continuous and decreasing function. However, a more accurate value is determined from the constant of lunisolar precession, without requiring the density to be known.

3. THE DYNAMICAL FLATTENING IN THE CASE OF THE ELLIPTICAL MOTION OF THE MOON

With a good approximation, the Poisson's equations can be integrated using the developments of the coordinates of the Sun and the Moon from the elliptical motion case. We use the theory of precession and nutation obtained by Serret in the case of the limitation of the series to the second order terms in eccentricity and inclination, theory exposed in Tisserand's treatise (Tisserand, 1891) and in Smart's book (Smart, 1953). If we consider the Earth's ellipsoid of inertia as an ellipsoid of rotation, then the force function that occurs in Poisson's equations has the expression

$$U = -\frac{3}{2}(C-A)\left(GM\frac{z^2}{\rho^5}+GS\frac{z_1^2}{\rho_1^5}\right), \tag{9}$$

where G is the gravitational constant, M is the Moon's mass, S is the Sun's mass, ρ and ρ_1 are the geocentric distances of the two bodies, and z, z_1 are the z - coordinates of these bodies in the system defined by the Earth's principal axes of inertia. Because the ratio of the mass of the Earth - Moon system to that of the Sun $(E+M)/S$ is of the order of 3×10^{-6}, then with sufficient accuracy $GS=n_1^2a_1^3$, where a_1 and n_1 denote the semi-major axis of the heliocentric orbit of the center of mass and the corresponding mean angular motion.

We then have

$$U = -\frac{3}{2}(C-A)n_1^2a_1^3\left(\frac{M}{S}\frac{z^2}{\rho^5}+\frac{z_1^2}{\rho_1^5}\right), \tag{10}$$

or, if a denotes the semi-major axis of the Moon's orbit, and ω denotes the angular velocity of the rotation of the Earth,

$$-\frac{U}{C\omega} = K\left[L\left(\frac{a}{\rho}\right)^3\left(\frac{z}{\rho}\right)^2+\left(\frac{a_1}{\rho_1}\right)^3\left(\frac{z_1}{\rho_1}\right)^2\right], \tag{11}$$

where

$$K = \frac{3}{2}\frac{C-A}{C}\frac{n_1^2}{\omega^2}, \tag{12}$$

$$L = \frac{M}{S}\left(\frac{a_1}{a}\right)^3. \tag{13}$$

Integrating Poisson's equations, one obtains for the constant of lunisolar precession the following expression

$$p = 2K\left[L\left(\frac{1}{2}+\frac{3}{4}e^2-\frac{3}{4}s^2\right)+\frac{1}{2}+\frac{3}{4}e_0^2\right]\cos\varepsilon_0, \tag{14}$$

e being the eccentricity of the Moon's orbit, e_0 - the eccentricity of the solar orbit at epoch, and $s=\sin c$, with c - the inclination of the lunar orbit to the ecliptic. Since $n_1^2a_1^3=G(S+E+M)\simeq GS$, it follows

$$L = \frac{M}{S}\left(\frac{a_1}{a}\right)^3=\frac{M}{E+M}\frac{G(E+M)}{GS}\left(\frac{a_1}{a}\right)^3=\frac{\mu}{1+\mu}\frac{n^2}{n_1^2}, \tag{15}$$

μ being the ratio of the mass of the Moon to that of the Earth and n - the sidereal mean motion of the Moon. We can write

$$p = H\cos\varepsilon_0\left(\mathrm{P}+\frac{\mu}{1+\mu}\mathrm{Q}\right), \tag{16}$$

where

$$H = \frac{C-A}{C}, \tag{17}$$

$$P = \frac{3n_1^2}{\omega}\left(\frac{1}{2}+\frac{3}{4}e_0^2\right), \tag{18}$$

$$Q = \frac{3n^2}{\omega}\left(\frac{1}{2}+\frac{3}{4}e^2-\frac{3}{4}s^2\right). \tag{19}$$

On the other hand, for the constant of nutation we have the expression

$$N = \frac{KLs}{\alpha}\left(1-\frac{1}{2}s^2+\frac{3}{2}e^2\right)\cos\varepsilon_0, \tag{20}$$

where $\alpha = |\mathrm{d}\Omega/\mathrm{d}t|$, Ω being the longitude of the ascending node of the lunar orbit. In the considered approximation for K and L, it follows

$$N = H\cos\varepsilon_0\frac{\mu}{1+\mu}\mathrm{R}, \tag{21}$$

where

$$R = \frac{3}{2}\frac{n^2 s}{\omega\alpha}\left(1-\frac{1}{2}s^2+\frac{3}{2}e^2\right). \tag{22}$$

For the constant of lunisolar precession and for the constant of nutation we shall use the values obtained by Newcomb at the epoch 1900, namely $p_N = 5037.08''$ per tropical century and $N = 9.210''$. But applying the correction of $1.10''$ per tropical century obtained by Fricke (1967) and the geodesic precession of $1.92''$ per tropical century, it follows $p = 5040.21''$ per Julian century. On the other hand, the obliquity of the ecliptic to the equator at the epoch 1900 was $\varepsilon_0 = 23^0 27' 8.26''$. For the mean motion of the Sun and the eccentricity of the orbit, we shall use the values of Newcomb's theory, namely $n_1 = 628.307590$ rad/cy, $e_0 = 0.01675104$. For the Moon, we shall use the values of Brown's theory, namely the eccentricity $e = 0.054900489$, the inclination of the orbit to the ecliptic $c = 5^0 8' 43.43''$, the sidereal mean motion $n = 8399.6850$ rad/cy, $\mathrm{d}\Omega/\mathrm{d}t = -33.757146$ rad/cy. We mention that the value n used by Brown is very close to the improved value adopted in the IAU (1964) System ($n = 8399.6848$ rad/cy). We shall use for ω the value adopted in 1967 by IUGG, namely $\omega = 230121.65297$ rad/cy (value practically identic with the value adopted in the IAU (2009) System), and for μ the value adopted by IAU (2009) System,

$\mu = 1/81.300568$. It follows

$$p = H\left(487126'' + 86367667''\frac{\mu}{1+\mu}\right), \tag{23}$$

$$N = H\cdot 231315''\frac{\mu}{1+\mu}. \tag{24}$$

At present, when the value of μ is sufficiently well known, H is determined from the relation (23). Since p is determined in fact with no more than six significant digits, it follows that H is obtained with the same number of significant digits. The relation (24) is used to calculate N in the rigid Earth's hypothesis. The comparison with the value obtained from observations shows to what extent this assumption is justified. Replacing the value of μ, we obtain $p = 1536544''H$/cy, $H = 0.00328022$, $N_c = 9.219''$. It follows that the difference between the observed value $N_o = 9.210''$, adopted in the first system of astronomical constants and in the IAU (1964) System, and the one calculated is $N_o - N_c = -0.009''$.

The obtained results can be improved considering the expression $G(S+E+M) = n_1^2a_1^3$, instead of the approximate expression $GS = n_1^2a_1^3$. Then U can be written

$$U = -\frac{3}{2}(C-A)GS\left(\frac{M}{S}\frac{z^2}{\rho^5} + \frac{z_1^2}{\rho_1^5}\right), \tag{25}$$

where

$$GS = \frac{GS(S+E+M)}{S+E+M} = \frac{n_1^2a_1^3}{1+(E+M)/S}. \tag{26}$$

On the other hand, M/S can be written as

$$\frac{M}{S} = \frac{M}{E+M}\frac{G(E+M)}{GS} = \frac{\mu}{1+\mu}\frac{n^2a^3}{GS} = \frac{\mu}{1+\mu}\frac{n^2a^3}{n_1^2a_1^3}\left(1+\frac{E+M}{S}\right). \tag{27}$$

It follows that only the expression of p is modified. It becomes

$$p = H\cos\varepsilon_0\left(\frac{\mathrm{P}}{1+(\mathrm{E}+\mathrm{M})/\mathrm{S}} + \frac{\mu}{1+\mu}\mathrm{Q}\right). \tag{28}$$

Considering for $(E+M)/S$ the value adopted in the IAU (1976) System, namely $1/328900.0$, we obtain $H = 0.003280222$. Therefore, because only six digits are significant, it follows that also in this case we have $H = 0.00328022$ with a good approximation. This value will be compared first of all with the value deduced from the theory of precession and nutation developed by Woolard.

4. THE DYNAMICAL FLATTENING DEDUCED FROM WOOLARD'S THEORY OF PRECESSION AND NUTATION

Woolard's theory is based on Newcomb's theory of the Sun and on Brown's theory of the Moon. The Earth is considered rigid and $A=B$. The precession angle is measured in the direct sense. From the series expansions that occur in Poisson's equations, for the constant of lunisolar precession and the constant of nutation follow the expressions

$$p = \frac{3GS(C-A)}{\omega_3 C \mathrm{arc}1''}(0.458887)+\frac{3GM(C-A)}{\omega_3 C \mathrm{arc}1''}(3422.54''\mathrm{arc}1'')^3, \quad (29)$$

$$N = \frac{GM(C-A)}{\omega_3 C \mathrm{arc}1''}\frac{(3422.54''\mathrm{arc}1'')^3}{|\mathrm{d}\Omega/\mathrm{d}t|}(0.041166), \quad (30)$$

where $3422.54''$ is the constant of Moon's sine parallax, and ω_3 is the component of the Earth's angular velocity of rotation along the polar principal axis of inertia. The component ω_3 practically coincides with ω.

The first expression can be written as

$$p = 0.458887 k_S + 0.455265 k_M, \quad (31)$$

where

$$k_S = \frac{3(C-A)}{C}\frac{S}{S+E+M}\frac{n_1^2}{\omega \mathrm{arc}1''}, \quad (32)$$

$$k_M = \frac{3(C-A)}{C}\frac{\mu}{1+\mu}\frac{1}{F_2^3}\frac{n^2}{\omega \mathrm{arc}1''}, \quad (33)$$

F_2 being the factor for the mean distance of the Moon. The value adopted in the IAU (1964) System is $F_2 = 0.999093142$. The second expression becomes

$$N = k_M \frac{0.041166}{|\mathrm{d}\Omega/\mathrm{d}t|}. \quad (34)$$

Obviously, k_S and k_M are the common factors of the solar terms, respectively of the lunar terms from the series that are involved in Poisson's equations. Substituting the values for n, n_1, ω and $(E+M)/S$, we obtain

$$k_S = 1061529.81'' H/\mathrm{cy}, \quad (35)$$

$$k_M = 190237867''\frac{\mu}{1+\mu}H/\mathrm{cy} \quad (36)$$

and

$$p = H\left(487122.23''+86608642''\frac{\mu}{1+\mu}\right). \quad (37)$$

Considering $p = 5040.21''$/cy and $\mu = 1/81.300568$, it follows

$$p = 1539468'' H/\mathrm{cy}, \tag{38}$$

$$H = 0.00327399, \tag{39}$$

$$k_S = 3475.4486''/\mathrm{cy}, \tag{40}$$

$$k_M = 7567.8320''/\mathrm{cy}. \tag{41}$$

By comparing the expression (37) with the corresponding expression (23) from the case of the elliptical motion of the Moon, an increase of the second term is observed.

On the other hand, taking $|\mathrm{d}\Omega/\mathrm{d}t| = 33.757146$ rad/cy, it results $N = 9.2288''$, a value greater than the value $N = 9.210''$.

We mention that the factors k_S and k_M can be calculated independently of H. Indeed, if p_S is the solar component of the constant of lunisolar precession and p_M is the lunar component, we have the relations

$$p = p_S + p_M, \tag{42}$$

$$\frac{p_M}{p_S} = \frac{0.455265}{0.458887} \frac{n^2}{F_2^3} \frac{\mu}{1+\mu} \frac{1+(E+M)/S}{n_1^2}, \tag{43}$$

from which result p_S and p_M, and therefore k_S and k_M. The values thus obtained practically coincide with those obtained previously.

Differentiating the function $H = H(p,\mu)$ given by the relation (37), we obtain for the variation δH the expression

$$\delta H = 6.49 \times 10^{-7} \delta p - 1.79 \times 10^{-1} \delta \mu. \tag{44}$$

Therefore, the quadratic mean error σ_H is given by the relation

$$\sigma_H^2 = 4.21 \times 10^{-13} \sigma_p^2 + 3.20 \times 10^{-2} \sigma_\mu^2. \tag{45}$$

Considering $\sigma_p = 0.1''$/cy and $\sigma_\mu = 10^{-7}$, we obtain $\sigma_H = 6.7 \times 10^{-8}$. Hence, from Woolard's theory it follows $H = 0.00327399 \pm 7 \times 10^{-8}$.

The value obtained for H allows to calculate the constant of lunisolar precession in the simplified case of the elliptical motion of the Moon. From the formula $p = 1536544'' H$/cy, established in Section 3, it follows $p = 5030.63''$/cy, therefore, a value less than the value deduced from observations, the deviation being of the order of $10''$ per Julian century. The values obtained for k_S and k_M differ from the ones deduced by Woolard, because he first determined k_M considering $N = 9.210''$. He obtained $k_M = 7552.4295''$/cy. Taking $p = 5037.19''$/cy, he found $k_S = 3484.15''$/cy. Hence, the common factor for the solar terms is greater, and the one for the lunar terms is less in comparison with the corresponding more accurate values obtained with the new values p and μ.

A new theory of the rotation of the Earth, based on Newcomb's theory of the

Sun and on Brown's theory of the Moon was subsequently developed by Kinoshita (1977). The Andoyer variables are adopted instead of Euler angles and the ecliptic of date as a reference plane. The Earth is considered as rigid triaxial and $H=[2C-(A+B)]/2C$. The author adopted Brown's tables improved by Eckert *et al.* (1966). This theory refines the theory of Woolard (1953). The effects due to the second order perturbations are also investigated.

Taking $p=5040.21''$/cy and $\mu=1/81.3007$, Kinoshita (1977) obtained

$$H = 0.00327395, \tag{46}$$
$$k_S = 3475.3977''/\mathrm{cy}, \tag{47}$$
$$k_M = 7567.7216''/\mathrm{cy}, \tag{48}$$
$$N = 9.2277''. \tag{49}$$

The value of H found by Kinoshita is close to the value obtained in this paper in the case of Woolard's theory. If in the expression of the constant of precession obtained by Kinoshita we don't consider the term that contains $\mathrm{d}\Omega/\mathrm{d}t$ (term of second order), we get $H=0.00327397$, a value even closer to $H=0.003399$. We must mention that, if the values n and n_1 from Eckert *et al.* (1966) are adopted, the calculated value for F_2 (with nine significant digits) coincides with the value used by Brown and adopted in the IAU (1964) System. We also mention that in a subsequent work, realized by Kinoshita and Souchay (1990), the term in $\mathrm{d}\Omega/\mathrm{d}t$ is no longer considered. We can conclude that in Brown's theory the most probable value of the dynamical flattening is $H=0.00327397\pm 7\times 10^{-8}$.

Remark. If the expression given by Newcomb for the constant of lunisolar precession is reduced to the epoch 1900 and the values $p=5040.21''$/cy, $\mu=1/81.300568$ are used, it follows $H=0.00327384$, a value close to the values obtained in the theories of Woolard and Kinoshita. This shows that Hansen's theory is close to Brown's theory.

5. VALUES OF THE DYNAMICAL FLATTENING OBTAINED USING THE EPOCH J2000

In the work of Kinoshita and Souchay (1990), the theory of rotation of the rigid Earth is based on the ELP 2000 theory for the motion of the Moon (Chapront *et al.*, 1983) and the VSOP 1982 theory (Bretagnon, 1982) for the motion of the planets. The new epoch J2000, adopted in the IAU (1976) System of the astronomical constants, is used and the calculations are carried to eight significant digits. For the new epoch, the authors have obtained for the constant of lunisolar precession the value $p=5040.9672''$/cy, using the constant of general precession $p'=5029.0966''$/cy and applying the correction $\delta p'=11.8706''$/cy (Kinoshita and Souchay, 1990). This correction is due to the direct and indirect effect of the planets, to the geodesic precession, etc.

On the other hand,

$$p = 3H\left(\frac{M}{M+E}\frac{n^2}{\omega}M_0 + \frac{S}{S+E+M}\frac{n_1^2}{\omega}S_0\right)\cos\varepsilon_0, \tag{50}$$

where $M_0 = 0.49765621$, $S_0 = 0.50021053$. Using the values $n = 8399.6847$ rad/cy, $n_1 = 628.307585$ rad/cy and $M/E = 1/81.30068$, $(E+M)/S = 1/328900.5$ from the paper of Kinoshita and Souchay (1990), we obtain $p = 1539711.9''H$ per Julian century and $H = 0.0032739678$, a value that is slightly different from the value $H = 0.0032739567$ given in the work (Kinoshita and Souchay, 1990). We mention that applying the same formula and the same numerical values, Williams (1994) obtained the value $H = 0.0032739677$, practically an identical value to the one obtained in our work.

But, if the calculations are carried to eight significant digits, the obtained value should be corrected, because in the formula for p does not intervene in fact the mean sidereal motion of the Sun, which contains the perturbation δn of the mean longitude at epoch, but the corrected value $n_{10} = n_1 - \delta n = 628.306623$ rad/cy, where $\delta n = 0.000962$ rad/cy (Bretagnon, 1982). This is the mean motion which satisfies Kepler's third law. By introducing the value n_{10} in the relation (50), it follows $p = 1539710.4''H$/cy and $H = 0.0032739708$, a value slightly greater than the previously calculated value $H = 0.0032739678$, the deviation being $\Delta H = 0.00000030$. Therefore, the seventh significant digit is modified. On the other hand, we mention that once with the more precise determination of the constant of lunisolar precession a more accurate value of the dynamical flattening will be obtained.

In the last two decades Lunar Laser Ranging (LLR), Very Long Baseline Interferometry (VLBI) and the study of proper motions of the stars (see, e.g., the important paper elaborated by Williams (1994) have led to the idea of reduction of the constant of general precession. Williams (1994) adopted for the constant of general precession the value $5028.77''$/cy at J2000, a value $0.3266''$/cy less than the value from IAU (1976) System. He obtained the correction $\delta p' = 11.8869''$/cy, $p = 5040.6569''$/cy and $H = 0.0032737634$.

Adopting the same value for the constant of general precession, namely $p' = 5028.77''$/cy, and applying the correction $\delta p' = 11.8745''$/cy, Souchay and Kinoshita (1996) obtained for the constant of lunisolar precession the value $p = 5040.6445''$/cy. Just like in the work of Kinoshita and Souchay (1990), the theory ELP 2000 for the motion of the Moon (Chapront *et al.*, 1983) and the theory VSOP 1982 (Bretagnon, 1982) for the motion of the planets are used. The values for n and n_1 are those from the previous work. But for masses were adopted the values from the ephemeris DE 245 (Newhall *et al.*, 1993), namely $M/E = 1/81.30059$ and $(E+M)/S =$

$1/328900.56$. For the terms M_0 and S_0 were obtained the values $M_0 = 0.49630353$ and $S_0 = 0.50021054$. We observe that in the formula (50) the value which intervenes is not the mentioned M_0 value, but $M_0/F_2^3 = 0.49765634$. It follows then $p = 1539713.5''H$/cy and the value $H = 0.0032737548$ given in Souchay and Kinoshita (1996). We must also mention that the value $F_2 = 0.999093142$ calculated for 1900 and adopted in the IAU (1964) System, the same as the one deduced by Brown, is valid also for J2000 and for the new values $n = 1732559343.18''$/cy and $n_1 = 129597742.26''$/cy, used by Souchay and Kinoshita.

Indeed, replacing in the series

$$F_2 = 1 - \frac{1}{6}m^2 + \frac{1}{3}m^3 + \frac{407}{2304}m^4 - \frac{67}{288}m^5 \tag{51}$$

$$- \frac{45293}{41472}m^6 - \frac{8761}{6912}m^7 - \frac{4967441}{7962624}m^8 + \frac{14829273}{39813120}m^9 + ... \tag{52}$$

the value

$$m = \frac{n_1}{n - n_1} = 0.080848937483, \tag{53}$$

we obtain $F_2 = 0.9990931418 \simeq 0.999093142$. Hence, $F_2(1900) = F_2(2000)$. As it has been previously shown, to obtain eight significant digits it is necessary that the mean sidereal motion of the Sun $n_1 = 129597742.26''$/cy =628.307585 rad/cy be replaced by $n_{10} = 628.306623$ rad/cy. It follows then $p = 1539712.5''H$/cy and $H = 0.0032737569$. We also mention that Bretagnon *et al.* (1997), using the same value for the constant of general precession, obtained $H = 0.0032737671$, a value close to that obtained by Williams ($H = 0.0032737634$), respectively Souchay and Kinoshita ($H = 0.0032737569$). By comparing the three values, we deduce $H = 0.00327376 \pm 2 \times 10^{-8}$, if we limit ourselves to six significant digits and adopt $\sigma_p = 0.01''$/cy, $\sigma_\mu = 10^{-7}$.

Subsequently, the value of the constant of general precession was slightly increased. Thus, in the IAU 2000 precession - nutation model (Mathews *et al.*, 2002) was adopted the value $5028.7962''$/cy. It was obtained $H = 0.0032737949$. In the work elaborated by Capitaine *et al.* (2003) is considered the same value for the constant of general precession and practically the same value for the dynamical flattening is obtained, namely $H = 0.0032737945$. On the other hand, Fukushima (2003) estimated the constant of general precession at J2000 as $(5028.7955 \pm 0.0003)''$/cy and adopted for the geodesic precession the value $p_g = (1.9196 \pm 0.0003)''$/cy. He obtained $H = 0.0032737804 \pm 3 \times 10^{-10}$.

If the correction $\delta p' = 11.8745''$/cy found by Souchay and Kinoshita (1996) is applied to the value of the constant of general precession $p' = 5028.7962''$/cy, the value $p = 5040.6707''$/cy for the constant of lunisolar precession is obtained. Using the formula previously established, namely $p = 1539712.5''H$/cy, it follows

$H = 0.0032737740$, a value greater than the value obtained before. Similarly, if the value of the correction $\delta p' = 11.8869$/cy obtained by Williams (1994) is applied and the same formula is used, it is obtained $H = 0.0032737820$. It results that the transition from the value $5028.7700''$/cy to the new values led to a slight increase of the dynamical flattening.

In the new IAU (2009) System of astronomical constants were adopted for precession the values proposed by Working Group on Precession and the Ecliptic (Hilton *et al.*, 2006), namely $5028.796195''$ per Julian century for the rate of general precession in longitude and $5038.481507''$ per Julian century for the rate of general precession of the equator in longitude. For more clarity, the denominations of lunisolar precession and planetary precession were replaced by *precession of the equator* and *precession of the ecliptic*, proposed by Capitaine *et al.* (2003). Because the adopted value for constant of general precession practically coincides with the value adopted in the IAU 2000 precession-nutation model, and in this model the dynamical flattening is obtained with great accuracy, one can consider that the value $H = 0.003273795$ is the most probable value of the dynamical flattening, if the value is limited to seven significant digits. This value is close to the value $H = 0.00327378$ discussed before.

We remark that the value of the dynamical flattening derived from astronomical observations is greater than the value deduced from geophysical considerations. In the paper of Dehant and Capitaine (1997), dedicated to a comparison between the two values, one considers that the difference is probably due to the fact that the hypothesis regarding the hydrostatic equilibrium of the Earth is not fulfilled. In the paper is presented the value $H = 0.003240$ obtained for the Preliminary Reference Earth Model (PREM), elaborated by Dziewonski and Anderson. In this model, considering the Earth in hydrostatic equilibrium, one obtains the density and afterwards the principal moments of inertia. Similarly, from the value of J_2, in the hypothesis of hydrostatic equilibrium, one can compute the principal moments of inertia, and therefore the dynamical flattening. One obtains $H = 0.003273$. Evidently, the astronomical determination of the flattening is more precise.

6. ON THE VARIATION OF THE DYNAMICAL FLATTENING

In the last decades it was established that the second degree zonal harmonic coefficient J_2 of the geopotential presents secular and periodic variations. Because $Ma_e^2 J_2 = C - (A+B)/2 = HC$ (M being the mass of the Earth and a_e its equatorial radius), it results that H is also variable. On the other hand, $J_2 = -\sqrt{5}\,\overline{C}_{20}$, $\overline{C}_{20}$ being the normalized coefficient of degree 2 and order 0 of the geopotential. Therefore $H = -\sqrt{5} M a_e^2 \overline{C}_{20}/C$.

The variations of $\overline{C}_{20}$, *i.e.* of J_2 are obtained from orbit determination of artificial satellites. Bourda and Capitaine (2004), analyzing the C_{20} data obtained from positioning of satellites between 1985 and 2002, estimated the secular variation of H as $\mathrm{d}H/\mathrm{d}t = -7.4 \times 10^{-9}$/cy and therefore $\mathrm{d}J_2/\mathrm{d}t = -2.5 \times 10^{-9}$/cy (the epoch being J2000). On the other hand, the periodic variations are of the order of 10^{-10}. In the IAU 2009 System of astronomical constants was adopted the value $\mathrm{d}J_2/\mathrm{d}t = -3.0 \times 10^{-9}$/cy. This value is concordant with the value $\mathrm{d}J_2/\mathrm{d}t = (-3.4 \pm 0.6) \times 10^{-9}$/cy, obtained by Morrison and Stephenson (1997) from eclipse data over two millennia. Similar results were also obtained in the paper of Burša *et al.* (2008). In this paper is studied the influence of the variability of J_2 on the dynamical flattening. It is obtained $\mathrm{d}H/\mathrm{d}t = -8.45 \times 10^{-9}$/cy. Adopting $\mathrm{d}H/\mathrm{d}t = -7.9 \times 10^{-9}$/cy and $\Delta H = -7.9 \times 10^{-9} \Delta t$, we obtain that at the epoch when Hipparchus discovered the precession (150 B.C.) ΔH was 1.7×10^{-7} and the value of the dynamical flattening was $H = 0.00327379 + 1.7 \times 10^{-7} = 0.00327396$.

7. CONCLUSIONS

From the analysis made in this paper, it follows that in the approximation of the homogeneous Earth the dynamical flattening has the greatest value, namely $H = f - f^2/2 = 0.00334720$, f being the geometrical flattening. In the approximation of the elliptical motion of the Moon, using the constant of lunisolar precession $p = 5040.21''$ per Julian century, deduced from Newcomb's constant of lunisolar precession at epoch 1900, and the ratio of the mass of the Moon to that of the Earth $\mu = 1/81.300568$, adopted in the IAU (2009) System of astronomical constants, one obtains $H = 0.00328022$. In the case of Woolard's theory of precession and nutation, based on Newcomb's theory of the Sun and on Brown's theory of the Moon, using the same values for p and μ, we obtained $H = 0.00327399 \pm 7 \times 10^{-8}$. The value found is close to the value $H = 0.00327395$ obtained by Kinoshita (1977), who used Brown's theory improved by Eckert *et al.* (1966).

Once with the adoption of the new epoch J2000 in the IAU (1976) System and of the constant of general precession $5029.0966''$ per Julian century, the value of H was improved. Our analysis revealed that if the calculations are carried out to eight significant digits, then it is necessary that the mean sidereal motion of the Sun be corrected by the perturbation of the mean longitude at epoch. The application of this correction leads to the change of the seventh significant digit from the value of the dynamical flattening, more exactly to a slight increase of H. Thus, for example, the value $H = 0.0032739678$, obtained by applying the formula (50) to the theory of Kinoshita and Souchay (1990), becomes $H = 0.0032739708$. Moreover, as it was mentioned in Section 5, the application of this correction leads to obtaining the formula $p = 1539712.5''H$ per Julian century, which allows a good determination for

H. In the last decades, different values for the constant of general precession have been deduced. For all these values, we have slightly different values of the dynamical flattening. On the other hand, it is necessary to introduce the correction due to the perturbation of the mean longitude at epoch. Analyzing the values presented in Section 5 and taking into account the quadratic mean errors corresponding to the constant of lunisolar precession and to the ratio of the mass of the Moon to that of the Earth, namely $\sigma_p = 0.001''$/cy and $\sigma_\mu = 10^{-9}$, one can consider that the value $H = 0.003273795 \pm 1 \times 10^{-9}$ is sufficiently accurate to be used in order to determine the moments of inertia of the Earth and allows the determination with a good approximation of the factors k_S and k_M involved in the series of Poisson's equations. This value was adopted by IAU (Petit and Luzum, 2010).

REFERENCES

Bretagnon, P.: 1982, *Astron. Astrophys.*, **114**, 278.

Bretagnon, P., Rocher, P., and Simon, J.L.: 1997, *Astron. Astrophys.*, **319**, 305.

Bourda, G., Capitaine, N.: 2004, *Astron. Astrophys.*, **428**, 691.

Burša, M., Groten, E., and Šima, Z.: 2008, *Astron. J.*, **135**, 1021.

Capitaine, N., Wallace, P. T., and Chapront, J.: 2003, *Astron. Astrophys.*, **412**, 567.

Chapront-Touzé, M., Chapront, J.: 1983, *Astron. Astrophys.*, **124**, 50.

Dehant, V., Capitaine, N.: 1997, *Celest. Mech. Dyn. Astron.*, **65**, 439.

Eckert, W. J., Wallker, M. J., and Eckert, D.: 1966, *Astron. J.*, **71**, 314.

Erzhanov, Zh. S., Kalybaev, A. A.: 1975, *Dokl. Akad. Nauk SSSR*, **223**, 844.

Erzhanov, Zh. S., Kalybaev, A. A.: 1984, *General theory of the rotation of the Earth*, Nauka, Moscow.

Fricke, W.: 1967, *Astron. J.*, **72**, 1368.

Fukushima, T.: : 2003, *Astron. J.*, **126**, 494.

Hill, G. W.: 1893, *Astron. J.*, **13**, 1.

Hilton, J. L., Capitaine, N., Chapront, J. *et al.*: 2006, *Celest. Mech. Dyn. Astron.*, **94**, 351.

Kinoshita, H.: 1977, *Celest. Mech.*., **15**, 277.

Kinoshita, H., Souchay, J.: 1990, *Celest. Mech. Dyn. Astron.*, **48**, 187.

Mathews, P. M., Herring, T. A., and Buffet, B. A.: 2002, *J. Geophys. Res.*, **107**, B4, ETG 3-1-3-26.

Morrison, L.V., Stephenson, F. R.: 1997, *Contemporary Phys.*, **38**, 13.

Newcomb, S.: 1895, *The elements of the four inner planets and the fundamental constants of astronomy*, U.S. Govt. Printing Office, Washington, DC.

Newhall, X. X., Standish, E.M., and Williams, J. G.: 1993, DE245 ephemeris of the planets and Moon.

Oppolzer, T.: 1882, *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, Bd I, W. Engelmann, Leipzig.

Petit, G., Luzum, B. (eds.): 2010, *IERS Conventions (2010)*, Verlag des Bundesamts für Kartographie und Geodäsie, Frankfurt am Main.

Smart, W. M.: 1953, *Celestial Mechanics*, Longmans, Green and Co., London.

Souchay, J., Kinoshita, H.: 1996, *Astron. Astrophys.*, **312**, 1017.

Tisserand, F.: 1891, *Traité de mécanique céleste*, t II, Gauthier-Villars, Paris.

Williams, J. G.: 1994, *Astron. J.*, **108**, 711.

Woolard, E. W.: 1953, Astron. Pap. Am. Ephemeris, **15**, pt. 1.

Received on 8 September 2011