ON HOMOGRAPHIC SOLUTIONS AND CENTRAL CONFIGURATIONS OF THE N-BODY PROBLEM

FERENC SZENKOVITS **VASILE URECHE**

Babeş-Bolyai University,
Faculty of Mathematics and Computer Science
Astronomical Observatory Cluj-Napoca
Str. Cireşilor 19, RO-400487 Cluj-Napoca, Romania
E-mail: fszenko@math.ubbcluj.ro

Abstract. The concept of central configuration is important in the study of total collisions, in the relative equilibrium state of a rotating system or in the variation of the topological type of the energy and angular momentum invariant manifolds in the *n*-body problem, in close connections with homographic motions. In this paper by using the variation of he moment of inertia we characterize those particular solutions of the *n*-body problem, in which the bodies form a central configuration at any moment of the motion, which are not necessary supposed to be homographic solutions.

Key words: *n*-body problem, central configurations, homographic solutions

1. INTRODUCTION

This paper concerns an old problem, which arose in celestial mechanics: the problem of central configurations of the *n*-body problem and connections with homographic motions. The *n* barycentric position vectors x_i of the *n* bodies with masses m_i form a *central configuration*, if the force of gravitation acting on m_i at the moment of the given configuration is proportional to the mass m_i and to the barycentric position vector x_i. The notion of a central configuration was introduced by Laplace (1789).

There are several reasons why central configurations are of interest in celestial mechanics. If the masses are released from a central configuration with zero initial velocity, then all particles accelerate toward the origin in such a way that the configuration collapses homothetically. The result is a collision

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singularity. Simple collision orbits of this kind were the first explicitly known solutions of the 3-body problem (Euler 1767). These are not the only possible orbits which end in collision of all n particles, but it can be shown that for any such orbit, the configuration is very close to central configurations. The homothetical collapse play an essential role in the stellar evolution (the initial collapse of a cloud of dust particles, the final collapse of the star), where the number of particles (molecules, then atoms or nuclei and electrons) is so large and the particles are so tiny that the configuration usually is approximated with a continuum.

A planar central configuration also gives rise to a family of periodic solutions. The particles are released from the central configuration with initial velocities normal to their position vectors and with magnitudes proportional to their distances from the origin. Each particle will traverse an elliptical orbit as in the Kepler problem; moreover, the configuration remains similar to the initial configuration throughout the motion, varying only in size, i.e., the solutions are homographic. If the velocities are just large enough, the orbits will be circular. As the velocities tend to zero, the ellipses become more and more eccentric and the periodic solutions approach the collision solutions of the previous paragraph.

The central configurations also play a role in the study of the topology of the energy and angular momentum invariant manifolds of the n-body problem. It is known that in the three-body problem bifurcations in the topological type of invariant manifold occur at the levels which contain the circular periodic orbits mentioned above (Smale 1970a,b; McCord at al. 1998).

Finding all central configurations for an arbitrary number n of points is a difficult problem, which is still open. In the trivial case $n = 2$ all configurations of the two bodies are central configurations. We list the most important results known presently in this direction, for $n \geq 3$.

For $n = 3$ the only no collinear central configuration is formed by the vertices of an equilateral triangle (Lagrange 1873). For $n = 4$ the unique no coplanar configuration is given by the vertices of a regular tetrahedron (Dziobec 1900). The general approach to central configurations is also due to O. Dziobec (1900).

The collinear central configurations are described by the following theorem of Moulton (1910): each enumeration of the points uniquely determines a central configuration in which the points lie collinearly in the given order. Therefore, there are exactly $n!/2$ distinct collinear central configurations. For $n = 3$ there are exactly three such configurations, discovered by Euler (1767).

In the last years many authors studied different aspects related to the central configurations (e.g. Moeckel 1990, Llibre 1991, Meyer and Scmidt 1993, Casasayas et al. 1994, Albouy 1995, Moeckel and Simó 1995).

A given solution of the problem of n bodies is called *homographic* if the configuration formed by the n bodies moves in the inertial barycentric coordinate system in such a way as to remain similar to itself when the time varies.

The basic result connecting homographic solutions and central configurations can be formulated as follows (see Wintner, 1941): A solution of the n-body problem with given values m_i of the masses, is homographic if and only if the mass points form the same central configuration for every moment t. As also Wintner mentioned, if there exists a continuum of distinct central configurations for n given masses, it may occur that these mass points form a central configuration for every moment t in a suitable solution which is not homographic, since the central configuration might then vary with t.

In this study we investigate the variation of the moment of inertia, if are considered only those particular solutions of the n-body problem, in which the bodies form a central configuration at any moment of time. The studied solutions are not necessary homographic ones.

2. CENTRAL CONFIGURATIONS AND HOMOGRAPHIC SOLLUTIONS

The Newtonian n-body problem concerns the motion of n point particles with masses $m_1, \ldots, m_n$ ($m_i > 0$, $i = 1, \ldots, n$), in the Euclidean space $\mathbf{R}^3$, subjected to the mutual Newtonian attractions.

The *configuration space* is the 3n-dimensional manifold

$$M' = \left\{ x = (x_1, x_2, \ldots, x_n) \in \left(\mathbf{R}^3\right)^n \,\middle|\, x \notin \Delta \right\} = \left(\mathbf{R}^3\right)^n \setminus \Delta, \tag{1}$$

where

$$\Delta = \bigcup_{1 \le i < j \le n} \Delta_{ij}, \text{with}\, \Delta_{ij} = \left\{ x \in \left(\mathbf{R}^3\right)^n \,\middle|\, x_i = x_j \right\}, \qquad 1 \le i < j \le n \tag{2}$$

is the collision-ejection set. The *potential energy* $V \in C^\infty(M';\, \mathbf{R})$ of the problem is given by

$$V(x) = -\sum_{1 \le i < j \le n} \frac{m_i m_j}{\left\| x_i - x_j \right\|}, \tag{3}$$

Δ being the set of singularities of V.

The equations of the motion of the n-body problem are:

$$m_i \ddot{x}_i = -\operatorname{grad}_i V(x), \qquad i = 1, \ldots, n, \tag{4}$$

where grad_i is taken with respect to the usual Euclidean metric on $\mathbf{R}^3$ in the i-th factor of $\left(\mathbf{R}^3\right)^n$.

In order to simplify the problem, we fix the center of mass of the system at origin. In other words, we want to consider the linear manifold

$$M = \left\{ x \in M' \middle| \sum_{i=1}^{n} m_i x_i = 0 \right\}. \tag{5}$$

In the n-body problem the *moment of inertia* of the system

$$I(x) = \sum_{i=1}^{n} m_i \left\| x_i \right\|^2 \tag{6}$$

and the force function $U = -V$ are connected in the *Lagrange–Jacobi equation*:

$$\ddot{I} = 2U + 4h, \tag{7}$$

where $h \in \mathbf{R}$ is the *energy constant* in the n-body problem (Wintner, 1947).

A configuration $x \in M$ is called *central configuration* in the n-body problem, if there is some $\lambda \in \mathbf{R}$ such that

$$\operatorname{grad}_i U(x) = \lambda m_i x_i. \tag{8}$$

An immediate result says that the configuration $x \in M$ is a central configuration if and only if x is a critical point of the map $x \mapsto I(x)U^2(x) : M \to \mathbf{R}$, that is:

$$d\left(IU^2\right)(x) = 0, \tag{9}$$

(e.g. Wintner, 1947).

A given solution $x_i = x_i(t)$ of the problem of n bodies is called *homographic* if the configuration formed by the n bodies at a given t_0 moves in the barycentric coordinate system in such a way as to remain similar to itself when t varies. By this is meant that there exist a scalar $r = r(t)$ and an orthogonal 3-mtrix $\Omega = \Omega(t)$ such that for every i and t one has $x_i(t) = r(t)\Omega(t)x_i^0$, where x_i^0 denotes x_i at some initial $t = t^0$.

The more important result connecting homographic motions and central configurations can be formulated as follows (Wintner, 1947): A solution $x_i = x_i(t)$ of the problem of n bodies, with given values m_i of the masses, is homographic if and only if there exist two functions $r(t), \phi(t)$ and n initial position vectors x_i^0 by means of which $x_1(t), \ldots, x_n(t)$ can be represented in the form

$$x_i = r\,\Omega\, x_i^{\,0},\ \text{where}\ r = r(t),\ \Omega = \Omega(t) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

with

$$r(t^0) = r^0 = 1,\ (\mathrm{r} = \mathrm{r}(\mathrm{t}) > 0),\ \phi(t^0) = \phi^0 = 0, \tag{11}$$

where $r = r(t),\quad \phi = \phi(t)$ may be chosen as any solution of the Lagrangian equations

$$\ddot{r} - r\dot{\phi}^2 = -m^0 / r^2, \quad r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0\,, \tag{12}$$

belonging to

$$L = \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + m^0 / r, \tag{13}$$

and satisfying (11), while $x_1^{\,0},\ldots,x_n^{\,0}$ is any central configuration belonging to $m_1,\ldots,m_n$, and

$$m^0 = U^0 / I^0, \quad I^0 = \sum m_i \left|x_i^{\,0}\right|^2, \quad U^0 = \sum_{j\neq k} \frac{m_j m_k}{\left|x_j^{\,0} - x_k^{\,0}\right|}. \tag{14}$$

The conservation of energy and angular momentum of the system (12) tell us that

$$\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{m^0}{r} = h^0, \qquad r^2\dot{\phi} = \left|C^0\right|, \tag{15}$$

with $h^0 = h / I^0$ and $C^0 = C / I^0$, where *h* is the total energy and *C* the total angular momentum of the system. Eliminating $\dot{\phi}$ from equations (15), we have

$$\dot{r}^2 = 2h^0 + \frac{2m^0}{r} - \frac{\left|C^0\right|^2}{r^2}. \tag{16}$$

Introducing the moment of inertia *I* in place of *r*, by using $I = r^2 I^0$, equation (16) is equivalent with

$$\dot{I}^2 = 8hI + 8\sqrt{I^0}\,U^0\sqrt{I} - 4\left|C\right|^2. \tag{17}$$

3. VARIATION OF THE MOMENT OF INERTIA

Let now consider those solutions $x = x(t)$ of the equation (4) of the *n*-body problem, in which the configuration *x(t)* is a central configuration at any moment *t*. These are not necessary homographic solutions (see Wintner, 1947, p. 298). For these solutions condition (9) implies that IU^2 is constant, i.e.

$$IU^2 = I^0\left(U^0\right)^2. \tag{18}$$

Substituting U from (18) in the Lagrange-Jacobi equation (7) one obtains:

$$\ddot{I} - 2\frac{U^0\sqrt{I^0}}{\sqrt{I}} = 4h. \tag{19}$$

Multiplying equation (19) by IU^2 and integrating, we obtain the equivalent form

$$\dot{I}^2 = 8hI + 8\sqrt{I^0}U^0\sqrt{I} + 8cI^0, \tag{20}$$

where c is a constant.

We can observe that equation (20) describing the variation of the moment of inertia of solutions supposed to form a central configuration for any moment is the same, with the equation (17), which describes the variation of the moment of inertia in homographic solutions, a subset of the solutions have been here considered.

Introducing the new variable r, connected with I by $I = r^2 I^0$, equation (20) leads to

$$\dot{r}^2 = \frac{2hr^2 + 2U^0 r + 2c}{I^0 r^2}, \tag{21}$$

or

$$\frac{rdr}{\sqrt{hr^2 + U^0 r + c}} = \pm\sqrt{2}dt. \tag{22}$$

This equation can be integrated, and according to the sign of the energy h we have different cases:

(i) If $h < 0$ and $2I^0\left(U^0\right)^2 \le h\left(\left(\dot{I}^0\right)^2 - 8U^0 I^0 - 8hI^0\right)$, then the motion is not possible.

(ii) If $h < 0$ and $2I^0\left(U^0\right)^2 > h\left(\left(\dot{I}^0\right)^2 - 8U^0 I^0 - 8hI^0\right)$, then the variation of the moment of inertia is described by:

$$\sqrt{hI + U^0\sqrt{I} + c} + \frac{U^0}{2\sqrt{-h}}\arccos\frac{c - 4h\sqrt{I}}{\sqrt{\left(U^0\right)^2 - 4ch}} = \pm\sqrt{2}h\left(t - \tau_1\right), \tag{23}$$

where τ_1 is the moment, when the left hand side expression is zero. The variation of the moment of inertia is periodic (fig. 1). The whole system has an “elliptical” type motion.

(iii) If $h = 0$, then

$$\sqrt{U^0\sqrt{I} + c}\left(U^0\sqrt{I} - 2c\right) = \pm\frac{3\sqrt{2}}{2}\left(U^0\right)^2\left(t - \tau_2\right), \tag{24}$$

where τ_2 is the moment, when the left hand side expression is zero. The variation of the moment of inertia is unbounded (fig. 2). The whole system has a motion of “parabolic” type.

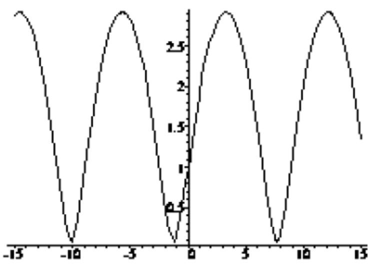

Figure 1: $h < 0$, ($I^0 = 1$, $\dot{I}^0 = 1$)

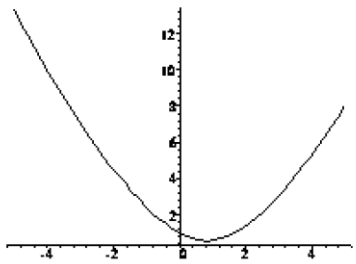

Figure 2: $h = 0$, ($I^0 = 1$, $\dot{I}^0 = -1$)

(iv) If $h > 0$, then

$$\sqrt{hI + U^0\sqrt{I} + c} - \frac{U^0}{2\sqrt{h}}\ln\left|\sqrt{I} + \frac{U^0}{2h} + \frac{1}{\sqrt{h}}\sqrt{hI + U^0\sqrt{I} + c}\right| = \pm 2h(t - \tau_3); \quad (25)$$

where τ_3 is the moment, when the left hand side expression is zero. The variation of the moment of inertia is boundless (fig. 3). The system has a motion of “hyperbolic” type.

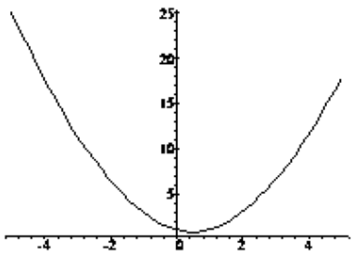

Figure 3: $h > 0$, ($I^0 = 1$, $\dot{I}^0 = -1$)

4. CONCLUSIONS AND ACCNOWLEDGMENT

Our results show that in this larger family of solutions, in which we suppose only that the configuration of the *n* bodies form all time a central configuration, there are only three types of possible motions, the same families of solutions as in the classical case of homographic motions. This result is immediate, if there are no continuum families of central configurations, including distinct central configurations, because in that case our family of solutions coincides with the homographic solutions. But the inexistence of continuum families of central configurations is only o not proved conjecture (Wintner, 1947; p. 282, §365). This result eventually is going to confirm that the conjecture is plausible.

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