THE TWO-BODY PROBLEM IN THE POINT MASS APROXIMATION FIELD. IV. SYMMETRIES

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Abstract. The only fields for which the correctness of the point-mass representation (Newton's theorem) can be proved are those featured by potentials of the form $A/r + Br^2$. The two-body problem in such a field is tackled from the only standpoint of symmetries. The motion equations, written in Cartesian or polar coordinates, present nice symmetries that form eight-element Abelian groups endowed with an idempotent structure. It is the same for McGehee-type coordinates that extend the phase space to collision or escape. All these groups are proved to be isomorphic. Expressed in Levi-Civita collision-regularizing coordinates, the vector field of the problem exhibits symmetries that form a sixteen-element group with the same characteristics.

Key words: celestial mechanics – two-body problem – point-mass approximation – symmetries.

1. INTRODUCTION

It is well-known that the only fields that allow rigorously the use of the point-mass representation (Newton's theorem) are those characterized by the Newtonian-type force, by the elastic-type force, or by a linear combination of these ones. Accordingly, the most general model of the two-body problem in such a field is featured by the potential $U = A/r + Br^2$, where r is the distance between particles, whereas A and B are real parameters.

Remark 1.1. The cases $A = 0$ (purely elastic-type force), $B = 0$ (attractive Newtonian gravitational force for $A > 0$, or repelling radiative force for $A < 0$), or even $A = 0 = B$ (degenerate case of the force-free field) are not of interest for us.

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However, for sake of completeness, such situations will be mentioned along this paper.

Of course, the celestial mechanics uses extensively the point-mass approximation, regardless to the field in which the motion is studied. This remains however a good approach in investigating the dynamics in fields of Manev-type, Schwarzschild-type, Einstein, Fock, Mücket-Treder, Reissner-Nordström, Schwarzschild - de Sitter, etc. (see, e. g., Mioc and Stavinschi 1999a and references therein). But we have to emphasize that the point-mass approximation is fully correct only for fields characterized by $A/r + Br^2$ potentials.

Some aspects of the problem were tackled by Mioc and Stavinschi (1999a, b, c). Using McGehee-type regularizing transformations, they described the local flows on the collision manifold and in its neighbourhood (Mioc and Stavinschi 1999b), as well as the equilibria of the problem (Mioc and Stavinschi 1999c).

This paper resumes the study of the two-body problem in the point-mass approximation field (which could also be called gravito-elstic field) from a single point of view: symmetries. After estabilishing the motion equations and the first integrals of energy and angular momentum in Cartesian coordinates (Section 2), we point out the eight-element Abelian group of symmetries that characterize the respective vector field (Section 3). Section 4 emphasizes an analogous group of symmetries for standard polar coordinates.

In Section 5 we point out the symmetries that features the motion equations expressed in collision-blow-up McGehee-type coordinates. Section 6 deals with the symmetries characteristic to the motion equations written in infinity-blow-up McGehee-type coordinates. These symmetries also form eight-element Abelian group.

Just for comparison purposes, Section 7 points out the symmetries exhibited by the motion equations written in Levi-Civita regularizing coordinates. They form a sixteen-element Abelian group.

Section 8 surveys the main results of the paper. The most important results states the isomorphic equivalence of all eight-element groups of symmetries pointed out in Section 3-6.

2. BASIC EQUATIONS

The two-body problem associated to the potential U can obviously be reduced to a central-force problem, by fixing one body (hereafter center) at the origin of coordinates and studying the relative motion of the other body (hereafter particles). The motion is planar and is described by the equations

$$\dot{\mathbf{q}} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}},$$
$$\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}}, \tag{2.1}$$

where $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2 \setminus \{(0,0)\}$ and $\mathbf{p} = (p_1, p_2) \in \mathbf{R}^2$ are the position (configuration) vector and the momentum vector of the particle, respectively. The Hamiltonian of the problem reads

$$H(\mathbf{q},\mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q}), \tag{2.2}$$

where $T(\mathbf{p}) = |\mathbf{p}|^2/2$ is the kinetic energy, whereas $-U(\mathbf{q}) = -A/|\mathbf{q}| - B|\mathbf{q}|^2$ is the potential energy.

Explicitly, the equations of motion read

$$\begin{aligned}
\dot{q}_1 &= p_1, \\
\dot{q}_2 &= p_2, \\
\dot{p}_1 &= \left[-\frac{A}{\left(q_1^2 + q_2^2\right)^{3/2}} + 2B \right] q_1, \\
\dot{p}_2 &= \left[-\frac{A}{\left(q_1^2 + q_2^2\right)^{3/2}} + 2B \right] q_2.
\end{aligned} \tag{2.3}$$

Remark 2.1. It is clear that the problem admits the first integral of energy $H(\mathbf{q},\mathbf{p}) = h = \text{constant}$, or explicitly

$$\frac{p_1^2 + p_2^2}{2} - \frac{A}{\left(q_1^2 + q_2^2\right)^{1/2}} - B\left(q_1^2 + q_2^2\right) = h, \tag{2.4}$$

where h is the energy constant. The first integral of angular momentum

$$L(\mathbf{q},\mathbf{p}) = q_1 p_2 - q_2 p_1 = C = \text{constant}, \tag{2.5}$$

where C is the angular momentum constant, also holds. However, these integrals will not play an important role in our search for symmetries.

3. SYMMETRIES IN CARTESIAN COORDINATES

Proposition 3.1. *The vector field (2.3) benefits of eight remarkable symmetries,* $S_i = S_i(q_1, q_2, p_1, p_2, t)$, $i = \overline{0,7}$, *as follows:*

$$
\begin{aligned}
S_0 &= (q_1, q_2, p_1, p_2, t) = I \text{ (identity)},\\
S_1 &= (q_1, q_2, -p_1, -p_2, -t),\\
S_2 &= (q_1, -q_2, p_1, -p_2, t),\\
S_3 &= (-q_1, q_2, -p_1, p_2, t),\\
S_4 &= (q_1, -q_2, -p_1, p_2, -t),\\
S_5 &= (-q_1, q_2, p_1, -p_2, -t),\\
S_6 &= (-q_1, -q_2, -p_1, -p_2, t),\\
S_7 &= (-q_1, -q_2, p_1, p_2, -t).
\end{aligned}
\tag{3.1}
$$

Proof. One sees immediately that equations (2.3) are invariant to the transformations described by (3.1).□

Proposition 3.2. *Out of the seven symmetries* S_i, $i = \overline{1,7}$, *only three are independent.*

Proof. Consider that S_1, S_2, S_3 are independent each other. One can easily check that

$$
\begin{aligned}
S_4 &= S_1 \circ S_2,\\
S_5 &= S_1 \circ S_3,\\
S_6 &= S_2 \circ S_3,\\
S_7 &= S_1 \circ S_2 \circ S_3.
\end{aligned}
\tag{3.2}
$$

Choosing arbitrarily three symmetries in $\{S_i \mid i = \overline{1,7}\}$ as reciprocally independent, a structure similar to (3.2) is recovered.□

Theorem 3.3. *The set* $G = \{S_i | i = \overline{0,7}\}$, *endowed with the composition law "*$\circ$*", forms a symmetric Abelian group with an idempotent structure.*

Proof. The composition table below

$\circ$	S_0	S_1	S_2	S_3	S_5	S_5	S_6	S_7

S_0	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S_1	S_1	S_0	S_4	S_5	S_2	S_3	S_7	S_6
S_2	S_2	S_4	S_0	S_6	S_1	S_7	S_3	S_5
S_3	S_3	S_5	S_6	S_0	S_7	S_1	S_2	S_5
S_4	S_4	S_2	S_1	S_7	S_0	S_6	S_5	S_3
S_5	S_5	S_3	S_7	S_1	S_6	S_0	S_4	S_2
S_6	S_6	S_7	S_3	S_2	S_5	S_4	S_0	S_1
S_7	S_7	S_6	S_5	S_4	S_3	S_2	S_1	S_0

can be easily constructed and checked. The Abelian character is obvious. As to the idempotent structure, it is clear that every element is its own inverse with respect to the composition law.

4. SYMMETRIES IN POLAR COORDINATES

As usual in central-force problems, w pas to polar coordinates via the transformations

$$
\begin{aligned}
r &= |\mathbf{q}|, \\
\theta &= \arctan(q_2/q_1), \\
u &= \dot{r} = (q_1 p_1 + q_2 p_2)/|\mathbf{q}|, \\
v &= r\dot{\theta} = (q_1 p_2 - q_2 p_1)/|\mathbf{q}|,
\end{aligned}
\tag{4.1}
$$

which also introduce the polar components of the velocity. Under these real analytic diffeomorphism, the vector field (2.3) turn to

$$
\begin{aligned}
\dot{\mathrm{r}} &= u, \\
\dot{\theta} &= \frac{v}{r}, \\
\dot{\mathrm{u}} &= \frac{v^2}{r} - \frac{A}{r^2} + 2Br, \\
\dot{\mathrm{v}} &= -\frac{uv}{r},
\end{aligned}
\tag{4.2}
$$

while the first integrals (2.4) and (2.5) become respectively

$$u^2 + v^2 = \frac{2A}{r} + 2Br^2 + 2h; \tag{4.3}$$

$$rv = C. \tag{4.4}$$

The results of Section 3 can be transposed in this new frame under the form of:

Proposition 4.1. *The vector field (4.2) has eight symmetries,* $\widetilde{S}_i = \widetilde{S}_i(r, \theta, u, v, t)$, $i = \overline{0,7}$, *as follows:*

$$\begin{aligned}
\widetilde{S}_0 &= (r, \theta, u, v, t) = I, \\
\widetilde{S}_1 &= (r, \theta, -u, -v, t), \\
\widetilde{S}_2 &= (r, -\theta, u, -v, t), \\
\widetilde{S}_3 &= (r, \pi - \theta, u, -v, t), \\
\widetilde{S}_4 &= (r, -\theta, -u, v, -t), \\
\widetilde{S}_5 &= (r, \pi - \theta, -u, v, -t), \\
\widetilde{S}_6 &= (r, \pi + \theta, u, v, t), \\
\widetilde{S}_7 &= (r, \pi + \theta, -u, -v, -t).
\end{aligned} \tag{4.5}$$

Proof. One can easily check that equations (4.2) are invariant to the transformations (4.5).□

Let us see what the symmetries (4.5) mean from a physical standpoint. Considering separately each argument of $\widetilde{S}_i$, *(t,-t)* means motion in the future/past; *(u,-u)* signifies outwards/inwards motion; *(v,-v)* means clockwards/counterclockwards motion; finally, $(\theta, -\theta)$, $(\theta, \pi - \theta)$, $(\theta, \pi + \theta)$ signify positions shifted each other by 2θ, $\pi - 2\theta$, and π, respectively. As to their combinations into symmetries, $\widetilde{S}_1$ corresponds to the reversibility of the flow: for each orbit there is another orbit with the same coordinates and with inverse velocities, all in reversed time. $\widetilde{S}_2$ shows that for every orbit there exists another orbit with inverse θ and v coordinates, and so on.

Imitating the proofs given in Section 3 to the corresponding results, we can state:

Proposition 4.2. *Out of the seven symmetries* $\widetilde{S}_i$, $i = \overline{1,7}$, *only three are independent.*

Theorem 4.3. *The set* $\widetilde{G} = \{\widetilde{S}_i | i = \overline{0,7}\}$ *, endowed with the composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure.*

5. SYMMETRIES IN COLLISION-BLOW-UP COORDINATES

The potential $U(\mathbf{q})$, the vector field (2.3), and the energy integral (2.4) have an isolated singularity at the origin $\mathbf{q} = (0,0)$. This singularity corresponds to a collision particle-center (e. g. Mioc and Stavinschi 2001). To remove it and to obtain regular equations of motion, we apply a chain of McGehee-type transformations of the second kind (McGehee 1974). The first step of these transformations, which was already performed in Section 4, consists of the real analytic diffeomorphism (4.1) that introduces the standard polar coordinates.

The next steps of the McGehee transformations consist of the real analytic diffeomorphisms

$$\begin{aligned} x &= r^{1/2} u, \\ y &= r^{1/2} v, \end{aligned} \tag{5.1}$$

which scale down the components of velocity, and

$$ds = r^{-3/2} dt \,, \tag{5.2}$$

which rescales the time. The vector field (4.2) becomes

$$\begin{aligned} r' &= xr, \\ \theta' &= y, \\ x' &= \frac{x^2}{2} + y^2 - A + 2Br^3, \\ y' &= -\frac{xy}{2}, \end{aligned} \tag{5.3}$$

whereas the first integrals (4.3) and (4.4) turn respectively to

$$x^2 + y^2 = 2A + 2Br^3 + 2hr; \tag{5.4}$$

$$r^{1/2} y = C. \tag{5.5}$$

Remark 5.1. In equations (5.3) we have denoted $(\cdot)' = d(\cdot)/ds$, keeping, by abuse, the same notation for the new functions of the timelike variable s.

Remark 5.2. In this way we have obtained regular equations of motion. The phase space was analytically extended to the boundary $r = 0$, which is invariant to the flow, because $r' = 0$ for $r = 0$. The singularity was replaced by the boundary manifold $M_0 = \{(r,\theta,x,y) \mid r = 0, \theta \in S^1, x^2 + y^2 = 2A\}$. The energy relation (5.4) also extends smoothly to this boundary.

Remark 5.3. The collision manifold M_0 is homeomorphic to a 2D cylinder or torus (pasting together the ends of the segment $S^1 = [0,2\pi]$) for $A > 0$, or to a circle for $A = 0$. For $A < 0$, $M_0 = \phi$ (the particle cannot collide with the center).

Remark 5.4. The collision manifold does not depend on the energy constant h, so every energy level shares this boundary.

Let us formally write $\hat{S}_i(r,\theta,x,y,s) = \hat{S}_i(r,\theta,u,v,t)$, $i = \overline{0,7}$. In this way, the following result can be stated without proof:

Proposition 5.5. *The vector field (5.3) benefits of eight symmetries,* $\hat{S}_i = \hat{S}_i(r,\theta,x,y,s)$, $i = \overline{0,7}$, *wholly similar to (4.5).*

Proposition 5.6. *Out of the seven symmetries,* $\hat{S}_i$, $i = \overline{1,7}$, *only three are independent.*

Theorem 5.7. *The set* $G_0 = \{\hat{S}_i | i = \overline{0,7}\}$, *endowed with the composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure.*

6. SYMMETRIES IN INFINITY-BLOW-UP COORDINATES

Another limit situation is the escape/capture ($r \to \infty$ in the future/past). This case also makes the motion equations (2.3), (4.2), or (5.3) singular, and the corresponding energy integrals, as well. To obtain regular equations of motion, we start from (5.3) and apply the McGehee-type transformation of the first kind (McGehee 1973)

$$\rho = r^{-1}, \tag{6.1}$$

which brings infinity at the origin. Then we use successively the McGehee-type transformations of the second kind (McGehee 1974)

$$\begin{aligned} \xi &= \rho^{3/2} x, \\ \eta &= \rho^{3/2} y; \end{aligned} \tag{6.2}$$

$$d\tau = \rho^{-3/2} ds. \tag{6.3}$$

Under these real analytic diffeomorphisms, the vector field (5.3) acquires the form

$$\begin{aligned} \frac{d\rho}{d\tau} &= -\rho\xi, \\ \frac{d\theta}{d\tau} &= \eta, \\ \frac{d\xi}{d\tau} &= -\xi^2 + \eta^2 - A\rho^3 + 2B, \\ \frac{d\eta}{d\tau} &= -2\xi\eta, \end{aligned} \tag{6.4}$$

where we kept, by abuse, the same notation for the new functions of the timelike variable τ. The first integrals (5.4) and (5.5) become respectively

$$\xi^2 + \eta^2 = 2A\rho^3 + 2B + 2h\rho^2; \tag{5.4}$$

$$\eta = C\rho^2. \tag{5.5}$$

Remark 6.1. In this way we have obtained regular equations of motion. The phase space was analytically extended to the boundary $\rho = 0$, which is invariant to the flow, because $d\rho / d\tau = 0$ for $\rho = 0$. The singularity at $\rho = 0$ was replaced by the boundary manifold $M_\infty = \{(\rho, \theta, \xi, \eta) \mid \rho = 0, \theta \in S^1, \xi^2 + \eta^2 = 2B\}$. The energy relation (6.5) also extends smoothly to this boundary.

Remark 6.2. The infinity manifold M_∞ is homeomorphic to a 2D cylinder or torus (see Remark 5.3) for $B > 0$, or to a circle for $B = 0$. For $B < 0$, M_∞ is the empty set (no escape/capture is possible).

Remark 6.3. The infinity manifold does not depend on the energy constant h, so every energy level shares this boundary.

Now, to point out the symmetries that characterize the vector field (6.4), let us formally write $S_i'(\rho,\theta,\xi,\eta,\tau)=\hat{S}_i(r,\theta,x,y,s)$, $i=\overline{0,7}$. We can state without proof:

Proposition 6.4. *The vector field (6.4) benefits of eight symmetries,* $S_i'=S_i'(\rho,\theta,\xi,\eta,\tau)$, $i=\overline{0,7}$, *wholly similar to (4.5).*

Proposition 6.5. *Out of the seven symmetries* S_i', $i=\overline{1,7}$, *only three are independent.*

Theorem 6.6. *The set* $G_\infty'=\{S_i' | i=\overline{0,7}\}$, *endowed with the composition law "$\circ$", forms a symmetric Abelian group with an idempotent structure.*

Remark 6.7. Observe, by (5.2) and (6.3), that the independent variable τ is just the physical time t. Also observe that we could regularize the motion equations for $r\to\infty$ by starting from the vector field (4.2). The motion equations obtained in this way also present eight symmetries, S_i'' $(i=\overline{0,7})$, say, which form a group G_∞'' with exactly the same properties as G_∞'.

7. SYMMETRIES IN LEVI-CIVITA COORDINATES

So far, to avoid singularities, we resorted to McGehee-type transformations. But there is a lot of regularizing transformations we could use. In the sequel, just for comparison purposes, we shall apply Levi-Civita's transformations

$$r=z^2,\quad \dot{r}=\frac{w}{z},\quad \dot{\theta}=\varphi; \tag{7.1}$$

$$d\sigma=z^{-3}dt. \tag{7.2}$$

to equations (4.2). The respective vector field turns to

$$\begin{aligned}
&\frac{dz}{d\sigma}=\frac{wz}{2},\\
&\frac{d\theta}{d\sigma}=\varphi z^3,\\
&\frac{dw}{d\sigma}=\frac{w^2}{2}+\varphi^2 z^6-A+2Bz^6,\\
&\frac{d\varphi}{d\sigma}=-2w\varphi,
\end{aligned} \tag{7.3}$$

where we maintained, by abuse, the notation for the new functions of the timelike variable σ.

Proposition 7.1. *The vector field (7.3) has sixteen symmetries, $\overline{S}_i=\overline{S}_i(z,\theta,w,\varphi,\sigma)$, $i=\overline{0,15}$, as follows:*

$$\begin{aligned}
\overline{S}_0&=(z,\theta,w,\varphi,\sigma)=I,\\
\overline{S}_1&=(z,\theta,-w,-\varphi,-\sigma),\\
\overline{S}_2&=(z,-\theta,w,-\varphi,\sigma),\\
\overline{S}_3&=(z,\pi-\theta,w,-\varphi,\sigma),\\
\overline{S}_4&=(-z,-\theta,w,\varphi,\sigma),\\
\overline{S}_5&=(z,-\theta,-w,\varphi,-\sigma),\\
\overline{S}_6&=(z,\pi-\theta,-w,\varphi,-\sigma),\\
\overline{S}_7&=(-z,-\theta,-w,-\varphi,-\sigma),\\
\overline{S}_8&=(z,\pi+\theta,w,\varphi,\sigma),\\
\overline{S}_9&=(-z,\theta,w,-\varphi,\sigma),\\
\overline{S}_{10}&=(-z,\pi+\theta,w,-\varphi,\sigma),\\
\overline{S}_{11}&=(z,\pi+\theta,-w,-\varphi,-\sigma),\\
\overline{S}_{12}&=(-z,\theta,-w,\varphi,-\sigma),\\
\overline{S}_{13}&=(-z,\pi+\theta,-w,\varphi,-\sigma),\\
\overline{S}_{14}&=(-z,\pi-\theta,w,\varphi,\sigma),\\
\overline{S}_{15}&=(-z,\pi-\theta,-w,-\varphi,-\sigma).
\end{aligned} \tag{7.4}$$

Proof. The invariance of equations (7.3) to the above transformations can be immediately verified.□

Imitating the proofs performed in the case of eight-element groups presented in the previous sections, we are in the position to state:

Proposition 7.2. *Out of the fifteen symmetries* $\overline{S}_i$, $i = \overline{1,15}$, *of the vector field (7.3), only four are mutually independent.*

Theorem 7.3. *The set* $\overline{G} = \{\overline{S}_i \mid i = \overline{0,15}\}$, *endowed with the same composition law "*$\circ$*" as* G, $\widetilde{G}$, G_0, G'_∞, G''_∞, *forms a symmetric Abelian group with an idempotent structure.*

Remark 7.4. Comparing the symmetries $\widetilde{S}_i$ $(i = \overline{1,3})$ with the symmetries which $\overline{S}_i$ $(i = \overline{1,3})$, it is clear that they have the same physical significance, respectively.

8. CONCLUSIONS

Surveying the results obtained in this paper, we can formulate:

Remark 8.1. The motion equations of our problem, expressed in Cartesian or polar coordinates, or in (collision-blow-up or infinity-blow-up) McGehee-type coordinates, present remarkable symmetries that form eight-element Abelian groups endowed with an idempotent structure.

Theorem 8.2. *The groups* G, $\widetilde{G}$, G_0, G'_∞ *and* G''_∞ *are isomorphic.*

Proof. Each of these groups is an Abelian group with three generators. According to the Fundamental Theorem of Abelian Groups, they are isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$.□

Remark 8.3. Theorem 8.2. is not a trivial results. Recall that the phase space corresponding to G_0 contains the supplementary boundary manifold M_0, whereas the one corresponding to G'_∞ (or G''_∞) contains the supplementary boundary manifold M_∞.

Remark 8.4. The group G is not only isomorphic, but also diffeomorphic to $\widetilde{G}$. The same relationship holds between the groups G'_∞ and G''_∞.

Remark 8.5. Among all these groups, $\widetilde{G}$ is the closest to the physical description of the motion, due to the use of both natural polar coordinates and physical time.

Remark 8.6. Using Levi-Civita coordinates for our problem, we find a sixteen-element Abelian group of symmetries, endowed with an idempotent structure.

The symmetries pointed out in this paper are of much help in understanding various aspects of the local flows or of the global flow. Indeed, for every orbit proved to exist, they point out the existence of many other orbits. Moreover, resorting to continuity, they are very useful to the study of perturbed problems depending on a small parameters ε, such that for $\varepsilon = 0$ one recovers the initial unperturbed problem.

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