# A SOLVABLE VERSION OF THE DIRECT PROBLEM OF DYNAMICS

GEORGE BOZIS [1], MIRA-CRISTIANA ANISIU [2], CRISTINA BLAGA [3]

[1] *University of Thessaloniki, Department of Theoretical Mechanics*
*54006 Thessaloniki, Greece*
*E-mail: gbozis@ccf.auth.gr*

[2] *Romanian Academy, "T. Popoviciu" Institute of Numerical Analysis*
*P.O. Box 68, RO-3400 Cluj-Napoca, Romania*
*E-mail: mira@ictp-acad.ubbcluj.ro*

[3] *Astronomical Institute of the Romanian Academy,*
*Astronomical Observatory Cluj-Napoca*
*Str. Cireşilor 19, RO-3400 Cluj-Napoca, Romania*
*E-mail: cpblaga@math.ubbcluj.ro*

*Abstract.* Given any two-dimensional potential $V(x, y)$, we find (if there exists, of course) without integration of the pertinent equation of motion a monoparametric family of orbits of the form $f(x, y) = y + h(x) = c$ (whose members are shifted paralelly to the $y$-axis) traced by a unit mass material point. The main tool to this end is the nonlinear in $\gamma = f_y / f_x$ second order partial differential equation of the inverse problem of Dynamics (Bozis, 1995). In general, a condition on the given potential is derived; in case this condition is fulfilled, the function $\gamma$ can be obtained as the common root of two polynomial equations. In certain particular cases (isotach orbits, one-dimensional potentials) the differential equations become ordinary and the solution is found to completion in a different manner.

*Key words:* monoparametric family of orbits – direct and inverse problem – isotach orbits.

## 1. INTRODUCTION

The two-dimensional inverse problem of Dynamics, as formulated by Szebehely (1974) aims to find all potentials $V(x, y)$ which can give rise to a given monoparametric family of curves $f(x, y) = c$ traced in the $xy$ Cartesian plane by a material point of unit mass. Szebehely's equation contains the energy dependence function $E = E(f(x, y))$, which must be known in advance.

A linear second order partial differential equation in $V$, involving only the potential and the family of orbits (not the energy dependence), was provided by Bozis (1984). This equation can be rearranged in order to face the direct problem, which consists in finding all the monoparametric families created by a given potential. The result is a nonlinear second order partial differential equation for a function $\gamma(x, y) = f_y / f_x$, related to the slope of the orbits of the family.

The difficulties in solving the nonlinear partial differential equation are reduced when considering orbits or/and potentials of a special type. Thus, Bozis and Stefiades (1993) and Bozis and Grigoriadou (1993) studied the case of homogeneous families of orbits produced by homogeneous potentials, reducing the problem to solving ordinary differential equations, while Bozis et al. (1997) obtained homogeneous families of orbits produced by inhomogeneous potentials by finding common roots of certain algebraic equations.

The problem we face here is of similar nature. Out of all families $f(x, y) = c$ which *a given potential* $V$ can create, to find (if there exist, of course) those with equation $f(x, y) = y + h(x) = c$, in other words: to find families with slope $\mathrm{d}y / \mathrm{d}x = 1/\gamma = h'(x)$ depending only on the variable $x$. Such families are actually plane Bertrand curves (Lipschutz, 1969). It turns out that, in general, the given potential must satisfy certain conditions. Then the families of orbits are obtained as common roots of a quadratic and a quartic equation. In some special cases the quadratic equation may vanish, so one has to deal only with the fourth order algebraic equation. There are potentials (one-dimensional or producing isotach orbits) for which it is necessary to work directly with the main equation (8) and to reduce the problem to solving ordinary differential equations.

## 2. THE SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

Suppose that the family of planar orbits

$$f(x, y) = c \tag{1}$$

is traced in the inertial frame $Oxy$ by a material point of unit mass under the action of a potential $V = V(x, y)$. The "slope" function for the family of orbits is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -1/\gamma$, where

$$\gamma = \frac{f_y}{f_x}. \tag{2}$$

The function $\gamma$ is determined in terms of (1) and also is uniquely determining the family (1), so it can be referred to as the family of orbits.

The nonlinear partial differential equation (Bozis, 1995) relating compatible potentials and families of orbits can be written as

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{3}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_x + \gamma V_y}(-\gamma_x V_x + (2\gamma\gamma_x - 3\gamma_y)V_y + \gamma(V_{xx} - V_{yy}) + (\gamma^2 - 1)V_{xy}).$$

### 3. SLOPE FUNCTIONS INVOLVING ONLY ONE VARIABLE

We consider the case when the slope function depends only on the variable $x$ (the corresponding case for $y$ can be handled similarly), hence

$$\gamma = \gamma(x), \tag{4}$$

excluding the trivial cases of null $\gamma$ or $\gamma'$. From now on $\gamma'$ denotes differentiation of $\gamma$ with respect to the unique variable $x$. In this case

$$f(x, y) = y + \int \frac{dx}{\gamma(x)} = c, \tag{5}$$

and the corresponding potential can be obtained by quadratures (Grigoriadou et al. 1999).

Family (5) has been studied from the viewpoint of programmed motion also, *i.e.*: To find compatible pairs of $\gamma(x)$ and $V(x, y)$ so that the members of (5) are lying inside a preassigned region of the $xy$ plane (Anisiu and Bozis, 2000). It has been shown that, for adequate preassigned regions, this can be effectuated and that, in general, there is a unique pair $\{\gamma(x), V(x, y)\}$.

Each of the above orbits (5) is traced with total energy $E$ given by Szebehely's equation which, in view of (4)–(5), reads

$$E = V - \frac{1 + \gamma^2}{2\gamma\gamma'}(V_x + \gamma V_y), \tag{6}$$

and allows for real orbits (5) in the region where $E \geq V$, *i.e.* in the region defined by the inequality

$$\frac{V_x + \gamma V_y}{\gamma\gamma'} \leq 0. \tag{7}$$

Our goal here is to check whether a given potential adopts families of orbits of the form (5). To this end we write the main equation (3) as

$$\frac{\gamma''\gamma}{\gamma'} = \frac{(2\gamma V_y - V_x)\gamma' + V_{xy}\gamma^2 + (V_{xx} - V_{yy})\gamma - V_{xy}}{V_x + \gamma V_y}. \tag{8}$$

Denoting by

$$\begin{aligned} P &= V_y\gamma + V_x \\ K &= 2V_y\gamma - V_x \\ L &= V_{xy}\gamma^2 + (V_{xx} - V_{yy})\gamma - V_{xy}, \end{aligned} \tag{9}$$

we write equation (8) as

$$\frac{\gamma''\gamma}{\gamma'} = \frac{(2\gamma V_y - V_x)\gamma' + V_{xy}\gamma^2 + (V_{xx} - V_{yy})\gamma - V_{xy}}{V_x + \gamma V_y}. \tag{10}$$

The expression $P$, originally factoring the left hand side of (3), can be identically equal to zero only if $K\gamma' + L = 0$. This is possible (for $\gamma = \gamma(x)$) only in the uninteresting case $V(x, y) = \text{const}$. In what follows we shall consider $P \neq 0$.

Because the left hand side of equation (10) does not depend on $y$, by differentiating it with respect to $y$ we obtain

$$(K_y P - K P_y)\gamma' = L P_y - L_y P. \tag{11}$$

Adopting for the partial derivatives the notation

$$V_{ij} = \frac{\partial^{i+j} V}{\partial x^i \partial y^j}, \quad i, j \in \{0,1,\ldots\},$$

we can write (11) in the form

$$B\gamma\gamma' = A_3\gamma^3 + A_2\gamma^2 + A_1\gamma + A_0, \tag{12}$$

where

$$\begin{aligned} A_3 &= -V_{12}V_{01} + V_{02}V_{11} \\ A_2 &= V_{20}V_{02} + V_{03}V_{01} - V_{12}V_{10} - V_{02}^2 + V_{11}^2 - V_{21}V_{01} \\ A_1 &= -V_{21}V_{10} + V_{12}V_{01} + V_{20}V_{11} + V_{03}V_{10} - 2V_{02}V_{11} \\ A_0 &= -V_{11}^2 + V_{12}V_{10} \\ B &= 3(V_{02}V_{10} - V_{01}V_{11}). \end{aligned} \tag{13}$$

*Comment*: If the potential $V$ is such that $B = 0$, in order to have a function $\gamma$ compatible with it, it is *necessary* that the right hand side of (12) is also zero. As seen from the last of equations (13), this case leads to $V_{02}/V_{01} = V_{11}/V_{10}$ with general solution

$$V = g(y + h(x)),\tag{14}$$

where $g$ and $h$ are arbitrary functions of their respective arguments.

To find a *necessary and sufficient* condition which the given potential (14) must satisfy in order to be compatible with a family of the form (5) is a straightforward, yet very laborious and tedious task. (The idea is to consider the cubic equation $A_3\gamma^3 + A_2\gamma^2 + A_1\gamma + A_0 = 0$ and the cubic derived from it after differentiating with respect to $y$, obtain the common root for $\gamma$ in terms of the potential and insert it into equation (3). Sixth order partial derivatives of $V(x, y)$ are expected to appear in the condition).

We prompt to say that potentials of the form (14) generate isotach orbits of the form (5). These orbits will be studied in Section 4.

Solving (12) for $\gamma\gamma'$, zeroing $(\gamma\gamma')_y$ and taking into account that $P \neq 0$, we are left with a quadratic equation in $\gamma$

$$H_2\gamma^2 + H_1\gamma + H_0 = 0,\tag{15}$$

the coefficients $H_2$, $H_1$ and $H_0 = -H_2$ containing derivatives of $V$ up to the fourth order. The explicit expressions for the coefficients are listed in the Appendix.

If, for the given potential, as generally expected, $H_2 \neq 0$ and $H_1 \neq 0$, then equation (15) represents a *necessary* condition so that a solution $\gamma = \gamma(x)$ exists. It is interesting to note that, in this case, we obtain a couple of mutually orthogonal families.

If only one of $H_2$ and $H_1$ is zero, we obtain $\gamma = 0$ or $\gamma = \pm 1$, *i.e.* $\gamma' = 0$, cases excluded from the beginning. It may happen that both $H_2$ and $H_1$ are zero, in which case relation (15) becomes an identity.

Using again equation (12), we obtain the expression of $\gamma$ as a ratio of polynomials in $\gamma$

$$\gamma' = \frac{A_3\gamma^3 + A_2\gamma^2 + A_1\gamma + A_0}{B\gamma}.\tag{16}$$

Differentiating with respect to $x$ both sides of (16), and substituting $\gamma'$ using (16) itself, we obtain $\gamma''$ as a ratio of polynomials in $\gamma$. This value of $\gamma''$, as well as $\gamma'$ given by (16), are then substituted in the main equation (10) and provide a quartic equation in $\gamma$

$$R_4\gamma^4 + R_3\gamma^3 + R_2\gamma^2 + R_1\gamma^1 + R_0 = 0.\tag{17}$$

The coefficients $R_i, i = 0,\dots,4$, depend on the derivatives of $V$ up to the fourth order and they are listed in the Appendix. Computer algebra packages, like Mathematica or Maple, are of great help in doing the calculations.

In case $H_2 \neq 0$ and $H_1 \neq 0$, the function $\gamma(x)$ has to satisfy the two equations (15) and (17). Therefore, the Sylvester determinant (Mishina and Proskuryakov, 1965) of the corresponding polynomials must be zero and this fact leads to a condition involving the derivatives of $V$ up to the fourth order.

If only one of $H_2$ and $H_1$ vanishes, we have no solution (with $\gamma, \gamma' \neq 0$) for our problem. If $H_2 = H_1 = 0$, we are left with the quartic equation, which has to be solved; then the solutions $\gamma = \gamma(x)$ (if any) are to be selected.

After finding the slope function, one obtains the family of orbits (5) compatible with the given potential.

Finally, if either $V_{01}$ or $V_{10}$ are zero, we have $B = 0$. Since the equations (15) and (17) were obtained for $B \neq 0$, we treat this case on first grounds in Section 5.

## 4. ISOTACH ORBITS

Let the given potential be of the form (14). Then all orbits (compatible with this potential) with $\gamma(x) = 1/h'(x)$ are *isotach*, *i.e.* they are traced with constant magnitude of the velocity. Indeed, according to (5) and (14), the potential is of the form $V = V(f(x, y))$ and is kept constant along each orbit $f(x, y) = c$. Thus, constant is also the kinetic energy of the moving unit mass. Isotach orbits have been studied by Szebehely (1963) and by Nahon (1964).

For isotach orbits it is

$$V_y = \gamma V_x . \tag{18}$$

Taking into account our restriction to orbits with $\gamma$ depending merely on $x$, from (18) we prepare $V_{xy}, V_{yy}$ and express them in terms of $V_x$ and $V_{xx}$. Then we insert these expressions into equation (3) and obtain the ordinary differential equation in $\gamma = \gamma(x)$

$$\frac{\gamma''}{\gamma} = \frac{2(\gamma^2 - 1)\gamma'}{\gamma(\gamma^2 + 1)} . \tag{19}$$

The meaning of the above equation is the following: if the function $h(x)$ in the given potential (14) leads to a function $\gamma(x) = \dfrac{1}{h'(x)}$ satisfying (19), then the orbits corresponding to this $\gamma(x)$ and produced by any potential of the form (14) (*i.e.* with arbitrary $g$) are isotach.

The general solution of (19) is

$$c_1 x + c_2 = \mathrm{arc}_0 \tan\gamma - \frac{\gamma}{1+\gamma^2}, \tag{20}$$

where $c_1, c_2$ are arbitrary constants and the subscript zero denotes the principal branch of the multivalued function $\arctan\gamma$.

The relation (20) does not permit us to express the function $\gamma(x)$ (and, consequently, the corresponding family $y + h(x) = c$) explicitly in closed form. Yet we know that $\gamma(x)$ can be defined as the inverse of the monotonic function $x = x(\gamma)$.

In spite of (and because of) the above obstacle, we can find the totality of isotach orbits of the form (5) in *parametric form* (Bozis and Borghero, 1998; Anisiu and Pal, 1999) by the following reasoning:

From (20) we get $x = x(\gamma, c_1, c_2)$, and from (5) $y = c - h(x) = c - H(\gamma)$, where $H(\gamma) = h(x(\gamma))$. We require, of course, that $\mathrm{d}h/\mathrm{d}x = 1/\gamma$, *i.e.* $(\mathrm{d}H/\mathrm{d}\gamma)(\mathrm{d}\gamma/\mathrm{d}x) = 1/\gamma$, which, by virtue of the general solution (20), leads to $\mathrm{d}H/\mathrm{d}\gamma = 2\gamma/[c_1(1+\gamma^2)^2]$, or

$$H = -\frac{1}{c_1(1+\gamma^2)} + c_0, \tag{21}$$

where $c_0$ is a new (superfluous) constant, which may be put equal to zero.

We conclude that, for any definite pair of constants $c_1, c_2$, the equations

$$x = \frac{1}{c_1}\left(\mathrm{arc}_0 \tan\gamma - \frac{\gamma}{1+\gamma^2} - c_2\right)$$
$$y = c + \frac{1}{c_1(1+\gamma^2)} \tag{22}$$

give, in parametric form, a monoparametric family of isotach orbits of the form (5) produced by any potential (14) $V = V(c)$. Varying along each orbit of the family is the parameter $\gamma$, whereas the parameter $c$ varies from orbit to orbit. One can check from (22) that indeed, as expected, $\mathrm{d}y/\mathrm{d}x = -1/\gamma$.

## 5. ONE-DIMENSIONAL POTENTIALS

As we mentioned at the end of Section 3, if the given potential is one-dimensional (either $V = V(x)$ or $V = V(y)$), the case needs to be treated separately. The question is if such a potential is compatible with a family of orbits of the form (4)–(5). It will be shown that the totality of these families can be found, in both subcases treated below:

a) For $V = V(x)$ equation (8) can be written as

$$\frac{\gamma''}{\gamma'} + \frac{\gamma'}{\gamma} - \frac{V_{xx}}{V_x} = 0. \qquad (23)$$

A first integration of (23) leads to

$$\frac{\gamma\gamma'}{V_x} = \frac{1}{2}k_0, \qquad (24)$$

where $k_0$ is a constant.

For any given potential $V = V(x)$, from (24) we obtain

$$\gamma(x) = \pm(k_0 V(x) + k_1)^{1/2}, \qquad (25)$$

where $k_1$ is a second constant, the orbits being real in the region $k_0 V(x) + k_1 \geq 0$. Therefore, in this case, the totality of orbits (a three-parametric set) is of the form (5). Each orbit is traced with total energy $E = -(1 + k_1)/k_0$ given by equation (6). Inequality (7) becomes $2/k_0 \leq 0$, so we have real orbits only for $k_0 < 0$, they being allowed in the region of the $xy$ plane defined by the strips $V(x) \leq -k_1/k_0$. (See also Example 4 of Section 6.)

b) For $V = V(y)$ equation (8) becomes

$$2\gamma' - \frac{\gamma\gamma''}{\gamma'} = \frac{V_{yy}}{V_y}. \qquad (26)$$

Evidently, each side of equation (26) has to be a constant, say $b_0 \neq 0$. (For $b_0 = 0$ it is easily seen that the potential $V = V_0 y$ produces the three-parametric family of parabolas $f(x, y) = y + c_0(x + c_1)^2 = c$ described in the entire plane for $V_0 c_0 > 0$ with energy $E = V_0 c + V_0/(4c_0)$.)

In what follows we shall omit in the potential an additive constant, but we shall include the multiplicative constant to take care of the allowed region of motion (Bozis and Ichtiaroglou, 1994). From $V_{yy}/V_y = b_0 \neq 0$ we find

$$V(y) = b_1 \exp(b_0 y) \qquad (27)$$

and from the differential equation $2\gamma - \gamma\gamma''/\gamma' = b_0$ we find its first integral $(b_0 - 2\gamma')/\gamma^2 = b_0/c_0$ (where $c_0 \neq 0, \pm\infty$ is a new constant), or

$$\frac{d\gamma}{c_0 - \gamma^2} = \frac{b_0}{2c_0}\,dx. \qquad (28)$$

(For $c = \pm\infty$ it is found that each potential (27) creates the family $y + (2/b_0)\ln|x + b_2| = c$, traced, with the energy $E = -(1/4)b_1 b_0^2 \exp(cb_0)$, everywhere in the $xy$ plane if $b_1 < 0$.)

Further integration of (26) depends on the sign of the constant $c_0$. Thus:

b₁) For $c_0 < 0$, from (28) we obtain $\gamma = \sqrt{-c_0}\tan\{[b_0/(2\sqrt{-c_0})]x + c_1\}$ and the corresponding three-parametric family

$$f(x, y) = y + \frac{2}{b_0}\ln\left|\sin\left(\frac{b_0}{2\sqrt{-c_0}}x + c_1\right)\right| = c. \tag{29}$$

The orbits (29) are traced with total energy $E = b_1(1 + c_0)\exp(b_0 c)$ and, according to the inequality (7), are traced in the entire $xy$ plane for $b_1 < 0$.

b₂) For $c_0 > 0$, from (28) we obtain $\gamma = \sqrt{c_0}\tanh[b_0/(2\sqrt{c_0}) + c_1]$ and the family

$$f(x, y) = y + \frac{2}{b_0}\ln\left|\sinh\left(\frac{b_0}{2\sqrt{c_0}}x + c_1\right)\right| = c, \tag{30}$$

parametrized by the constants $c_0, c_1, c$.

The total energy is $E = -b(1 + c_0)\exp(b_0 c)$ and the orbits are traced in the entire plane for $b_1 > 0$.

In both subcases, equations (29) and (30) are of the form (5) and represent the totality of orbits traced in the presence of the potential (27).

## 6. EXAMPLES

**Example 1.** *For the potential*

$$V(x, y) = \frac{x^4}{4} + y + \left(y - \frac{x^4}{8}\right)^3\left(x^6 + 4\right)$$

*we have* $H_2 \neq 0, H_1 \neq 0$; *the calculations show that the two equations* (15) *and* (17) *have, indeed, a common root. This is* $\gamma(x) = x^3/2$.

**Example 2.** *The potential*

$$V(x, y) = 8y^2 + 4x^4 y - x^8 - 6x^2$$

*gives* $H_2 = H_1 = 0$. *Equation* (15) *is an identity and the only suitable (i.e. compatible with the given potential) solution of the quartic* (17) *is* $\gamma(x) = x^3/2$.

*Comment*: We note that the potentials in Examples 1 and 2 produce the same family of orbits $y - 1/x^2 = c$, of the form (5). However, members of this family which are real orbits of each potential are lying in different regions of the $xy$ plane (Erdi and Bozis, 1994). For the potential in Example 1 the allowed region is given by $y \le x^4/8 - 1/\sqrt[3]{4x^2}$, each orbit being described with total energy $E = c$, while for that in Example 2 the allowed region is given by $y \le x^4/4 + 1/2x^2$, the total energy being $E = 8c^2$.

**Example 3.** *For the potential*

$$V(x, y) = (4x^2 + 1)(x^2 - y)$$

*we have also* $H_2 = H_1 = 0$, *but since* $R_4 = 0$, *equation* (17) *is now of the third order and has compatible root* $\gamma(x) = 2x$ *corresponding to the family*

$$y + (1/2)\ln|x| = c.$$

*The orbits of the family are described in the region* $y \ge x^2$ *with energy* $E = 0$.

**Example 4.** *For* $V(x) = -1/x^2,\ x > 0$, *from the formula* (25) *we obtain* $\gamma = \pm(k_1 - k_0/x^2)^{1/2}$ *and*

$$f(x, y) = y \pm (1/k_1)(k_1x^2 - k_0)^{1/2} = c. \tag{31}$$

*Inequality* (7) *gives that* $2/k_0 \le 0$, *i.e.* $k_0$ *must be negative so that* (31) *represents real orbits. If* $k_1 > 0$, *the family* (31) *is defined everywhere; if* $k_1 < 0$, *the family is defined in the strip* $0 < x < (k_0/k_1)^{1/2}$. *In both cases the total energy is* $E = -(1 + k_1)/k_0$.

**Example 5.** *For the potential*

$$V(x, y) = y^3 + x^4y^2 + x^2$$

*there is no compatibility with a family of orbits with* $\gamma = \gamma(x)$, *because the quadratic and the quartic have no common solution.*

## 7. APPENDIX

$$H_2 = -V_{01}V_{13}V_{11} + V_{01}V_{12}^2 - V_{12}V_{03}V_{10} + V_{13}V_{02}V_{10} + V_{03}V_{11}^2 - V_{12}V_{02}V_{11}$$

$$\begin{aligned} H_1 = & -V_{01}V_{03}V_{12} + V_{01}V_{12}V_{21} - V_{01}V_{22}V_{11} + V_{01}V_{04}V_{11} - V_{03}V_{02}V_{11} - V_{04}V_{02}V_{10} \\ & - V_{20}V_{02}V_{12} + V_{03}^2V_{10} - V_{21}V_{03}V_{10} + V_{22}V_{02}V_{10} + V_{20}V_{03}V_{11} + V_{02}^2V_{12} \end{aligned}$$

$$H_0 = -H_2$$

$$\begin{aligned} R_4 = & \ 4V_{20}V_{02}V_{12}V_{01} - V_{11}V_{02}V_{03}V_{01} + 3V_{21}V_{02}^2V_{10} - 5V_{02}V_{11}V_{12}V_{10} - 3V_{22}V_{01}V_{02}V_{10} \\ & + V_{21}V_{02}V_{01}V_{11} + 5V_{02}V_{11}^3 + V_{02}^3V_{11} - 5V_{12}V_{01}V_{11}^2 + 3V_{22}V_{01}^2V_{11} + 5V_{12}^2V_{01}V_{10} \\ & + V_{03}V_{01}^2V_{12} - 4V_{21}V_{01}^2V_{12} - 4V_{11}V_{02}^2V_{20} - V_{02}^2V_{12}V_{01} \end{aligned}$$

$$\begin{aligned} R_3 = & \ 2V_{20}V_{11}^2V_{02} - V_{03}^2V_{01}^2 - 4V_{21}^2V_{01}^2 + 5V_{11}^4 - V_{02}^4 + 5V_{12}^2V_{10}^2 - 3V_{02}^2V_{11}^2 + 4V_{03}V_{11}V_{02}V_{10} \\ & + 5V_{02}^3V_{20} + 2V_{21}V_{11}V_{02}V_{10} - V_{20}V_{11}V_{12}V_{10} - 3V_{22}V_{10}^2V_{02} - 2V_{02}^2V_{12}V_{10} + 3V_{30}V_{02}^2V_{10} \\ & - 4V_{20}^2V_{02}^2 - V_{12}V_{10}V_{20}V_{02} + V_{03}V_{11}^2V_{01} + 3V_{31}V_{01}^2V_{11} - 4V_{21}V_{11}^2V_{01} + 3V_{22}V_{10}V_{01}V_{11} \\ & + 5V_{21}V_{01}^2V_{03} + 2V_{12}^2V_{01}^2 - 3V_{13}V_{01}^2V_{11} - 5V_{03}V_{10}V_{12}V_{01} + 2V_{21}V_{10}V_{12}V_{01} - 5V_{02}^2V_{21}V_{01} \\ & - 3V_{31}V_{01}V_{02}V_{10} + 3V_{13}V_{01}V_{02}V_{10} - 10V_{12}V_{10}V_{11}^2 + 2V_{02}^2V_{03}V_{01} + 3V_{02}V_{12}V_{01}V_{11} \\ & - 3V_{30}V_{02}V_{01}V_{11} + 8V_{21}V_{01}V_{20}V_{02} - 5V_{03}V_{01}V_{20}V_{02} \end{aligned}$$

$$\begin{aligned} R_2 = & -6V_{21}V_{02}^2V_{10} + 9V_{11}V_{02}^2V_{20} + 3V_{02}^2V_{12}V_{01} + 6V_{21}V_{01}^2V_{12} - 3V_{03}V_{01}^2V_{12} - 3V_{12}^2V_{01}V_{10} \\ & - 3V_{22}V_{01}^2V_{11} + 6V_{12}V_{01}V_{11}^2 + 6V_{02}V_{11}V_{12}V_{10} - 6V_{20}V_{02}V_{12}V_{01} + 3V_{11}V_{02}V_{03}V_{01} \\ & - 3V_{21}V_{02}V_{01}V_{11} + 6V_{21}V_{10}^2V_{12} - 6V_{21}V_{10}V_{11}^2 + 6V_{03}V_{10}V_{11}^2 - 3V_{31}V_{10}^2V_{02} + 3V_{13}V_{10}^2V_{02} \\ & - 6V_{03}V_{10}^2V_{12} - 3V_{21}^2V_{10}V_{01} - 3V_{30}V_{11}^2V_{01} - 6V_{20}V_{11}V_{12}V_{10} + 3V_{22}V_{01}V_{02}V_{10} \\ & + 3V_{30}V_{11}V_{02}V_{10} + 3V_{21}V_{10}V_{20}V_{02} + 3V_{20}V_{11}V_{21}V_{01} - 3V_{03}V_{20}V_{01}V_{11} - 9V_{02}V_{11}^3 \\ & - 3V_{02}^3V_{11} + 6V_{20}V_{11}^3 \end{aligned}$$

$$\begin{aligned} R_1 = & \ 10V_{21}V_{10}V_{11}^2 + V_{02}^2V_{12}V_{10} + V_{03}V_{11}^2V_{01} + 3V_{22}V_{10}^2V_{02} + 2V_{21}V_{11}^2V_{01} + 3V_{20}V_{11}^2V_{02} \\ & - 2V_{03}V_{10}^2V_{21} - 5V_{11}^4 - 2V_{12}^2V_{01}^2 - 5V_{12}^2V_{10}^2 + V_{20}^2V_{11}^2 - 3V_{02}^2V_{11}^2 + V_{21}^2V_{10}^2 + V_{03}^2V_{10}^2 \\ & - V_{03}V_{11}V_{02}V_{10} + 5V_{02}V_{12}V_{01}V_{11} - 3V_{22}V_{10}V_{01}V_{11} - 5V_{21}V_{11}V_{02}V_{10} - V_{12}V_{10}V_{20}V_{02} \\ & - 4V_{20}V_{11}V_{12}V_{01} - 2V_{03}V_{10}V_{12}V_{01} + 2V_{03}V_{10}V_{20}V_{11} + 5V_{21}V_{10}V_{12}V_{01} - 2V_{21}V_{10}V_{20}V_{11} \end{aligned}$$

$$R_0 = -(V_{11}^2 - V_{12}V_{10})(V_{02}V_{11} + V_{20}V_{11} - 2V_{12}V_{01} + V_{03}V_{10} - V_{21}V_{10})$$

## REFERENCES


Anisiu, M.-C., Bozis, G.: 2000, *Inverse Problems*, **16**, 19.

Anisiu, M.-C., Pal, A.: 1999, *Inverse Problems*, **15**, 135.

Bozis, G.: 1984, *Astron. Astrophys.*, **134**, 360.

Bozis, G.: 1995, *Inverse Problems*, **11** (special issue), 687.

Bozis, G., Grigoriadou, S.: 1993, *Celest. Mech.*, **57**, 461.

Bozis, G., Ichtiaroglou, S.: 1994, *Celest. Mech.*, **58**, 371.
Bozis, G., Stefiades, A.: 1993, *Inverse Problems*, **9**, 233.
Bozis, G., Anisiu, M.-C., Blaga, C.: 1997, *Astron. Nachr.*, **318**, 313.
Bozis, G., Borghero, F.: 1998, *Inverse Problems*, **14**, 41.
Erdi, B., Bozis, G.: 1994, *Celest. Mech.*, **60**, 421.
Grigoriadou, S., Bozis, G., Elmabsout, B.: 1999, *Celest. Mech.*, **74**, 211.
Lipschutz, M. M.: 1969, *Differential Geometry*, Schaum's Outlines Series, McGraw-Hill Co.
Mishina, A. P., Proskuryakov, I. V.: 1965, *Higher Algebra*, Pregamon Press, Oxford, London.
Nahon, F.: 1964, *C. R. Acad. Sci. Paris*, **259**, 65.
Szebehely, V.: 1963, *Arch. Rat. Mech. Anal.*, **13**, 192.
Szebehely, V.: 1974, in E. Proverbio (ed.), *Proc. Int. Meeting on Earth's Rotation by Satellite Observations*, Bologna, Univ. of Cagliari, 31.

*Received on 17 October 2000*