THE EQUILIBRIUM OF POLYTROPES IN A POLOIDAL MAGNETIC FIELD

CRISTINA BLAGA

Astronomical Institute of the Romanian Academy
Astronomical Observatory Cluj-Napoca
Str. Cireşilor, 19, 3400, Cluj-Napoca, Romania

Abstract. In this article we study the equilibrium of polytropes in a poloidal magnetic field. We see that the magnetic field is the solution of a nonhomogeneous Euler equation for which we write the general solution. It depends on the Lane–Emden function. So we determined the exact solution for $n = 0, 1$. To study the relation between the magnetic and the polytropic index we use the first order Padé approximants for the Lane–Emden function.

Key words: stellar structure, magnetic field.

1. INTRODUCTION

The detection of the magnetic field at the stellar surface (see Babcock, 1958) raised the question of its role in the equilibrium of the star. Earlier, Chandrasekhar and Fermi (1953) found, using the virial theorem, that there is a maximum value for the magnetic field above which the equilibrium is lost. The measured values are below it, so the stars are in equilibrium. But, the problem of the distribution of the magnetic field in the star raises a quite complicated mathematical problem. An analytical solution of the problem could be found only if we make some supplementary hypothesis. We shall presume, based on the photometric observations (Borra and Landstreet, 1980), that at the stellar surface the magnetic field can be approximated by a dipole with the center in the center of the star and so the field is weak and has an axial symmetry. We also consider that the star is a complete polytrope with a given polytropic index n. In this case to obtain the distribution of the magnetic field in the star we have to solve a singular Sturm-Liouville problem (see Roxburgh, 1966). But in the particular case of the poloidal magnetic field (see the decomposition proposed by Lüst and Schlüter, 1954), the problem can be easily solved, because it yields a nonhomogeneous Euler equation. Its general solution will depend on the Lane–Emden function. We will see that the first order Padé approximants, proposed by Pascual (1977), provide us a good approximation of the problem and its analytic form allows us a qualitative interpretation.

The equilibrium of a polytrope in a magnetic poloidal field was studied by Monaghan (1965). He found the differential equation that provides the magnetic field, solved it numerically for different polytropic indices and discussed the influence of the polytropic index n on the distribution of the magnetic field. Further, we will show that Monaghan's differential equation for the poloidal magnetic field is equivalent to an Euler equation and write its general solution. The Lane–Emden function present in this general solution will be substituted by its first order Padé approximants (Pascual, 1977), fact that permits us a comparison between this solution and a numerical one.

2. BASIC EQUATIONS

To describe the equilibrium in a poloidal magnetic field we will use the equation of the hydromagnetic equilibrium, the Poisson equation, the Ampère law, the magnetic monopole equation and the polytropic relation, respectively:

$$\frac{\nabla P}{\rho} = -\nabla\phi + \frac{\boldsymbol{j}\times\boldsymbol{H}}{c\rho}, \tag{1}$$

$$\nabla^2\phi = 4\pi G\rho, \tag{2}$$

$$\operatorname{curl}\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}, \tag{3}$$

$$\operatorname{div}\boldsymbol{H} = 0, \tag{4}$$

$$P = K\rho^{1+\frac{1}{n}}, \tag{5}$$

in which the notations are usual. Supposing that the star has axial symmetry and using the spherical coordinates (r,θ,ϕ) with $r=0$ in the center of the star and $\theta=0$, the symmetry axis, we obtain that $\dfrac{\partial}{\partial\phi}=0$. Having in mind the representation of the general solution of (4) (see Chandrasekhar, 1961) and the simplifications due to the hypothesis of axial symmetry and poloidal magnetic field, we will obtain that $\boldsymbol{H} = (H_r, H_\theta, 0)$, with:

$$H_r = \frac{1}{r^2\sin\theta}\frac{\partial S}{\partial\theta} \quad ; \quad H_\theta = -\frac{1}{r\sin\theta}\frac{\partial S}{\partial r}. \tag{6}$$

Taking curl of equation (1) and using (3) we will obtain:

$$\operatorname{curl}\left(\frac{\boldsymbol{H}\times\operatorname{curl}\boldsymbol{H}}{\rho}\right)=0, \tag{7}$$

relation that will be used further to express the dependence of the magnetic field on the radius. We shall assume, in the first approximation, that the magnetic field is weak, so it presence will not modify the mass density distribution in the star. So, $\rho=\rho_0(r)$, where ρ_0 is known from the equilibrium of an unperturbed polytrope.

We will consider that the exterior magnetic field is dipolic and in the interior it is expressed by:

$$S(r,\theta)=A(r)\sin^2(\theta). \tag{8}$$

To facilitate the evaluation of the magnetic field we introduce the following transformations:

$$r=a\xi \quad , \quad \rho_0=\rho_c\theta_n^n \quad , \quad a=\frac{K(n+1)\rho^{\frac{1}{n}-1}}{4\pi G} \quad , \quad A=D\rho_c a^4\gamma_n \tag{9}$$

in which we recognize the Emden variables (ξ,θ_n), introduce a new dimensionless function $\gamma_n(\xi)$ proportional to the magnetic field and use the constant of integration D (see Roxburgh, 1966). This substitution allows us to build the dimensionless form of (7). So, using (6), (8) and (9) in (7), we get the following nonhomogeneous second order differential equation:

$$\frac{\partial^2\gamma_n}{\partial\xi^2}-\frac{2\gamma_n}{\xi^2}=-\theta_n^n\xi^2 \tag{10}$$

where θ_n is the Lane–Emden function of index n, i.e. the solution of the Lane–Emden equation of order n:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta_n}{\mathrm{d}\xi}\right)=-\theta_n^n, \tag{11}$$

The boundary conditions for the equation (10) are:

$$\gamma_n=\gamma_n'=0 \quad \text{in} \quad \xi=0 \quad \text{and} \quad \xi\frac{\mathrm{d}\gamma_n}{\mathrm{d}\xi}+\gamma_n=0 \quad \text{in} \quad \xi=\xi_1, \tag{12}$$

The first condition says that the magnetic field must be finite in the center of the star and the second in $\xi=\xi_1$ (with ξ_1 the polytropic radius, i.e. the first zero of the Lane–Emden function) reflects that at the surface of the star there is no discontinuity in the magnetic field.

3. THE GENERAL SOLUTION OF EQUATION (10)

The equation (10) is a nonhomogeneous Euler equation. With the transformation $\xi = e^t$ it becomes an equation with constant coefficients:

$$\frac{d^2\widetilde{\gamma}}{dt^2} - \frac{d\widetilde{\gamma}}{dt} - 2\widetilde{\gamma} = -\widetilde{\theta}^n e^{4t}, \tag{13}$$

where $\widetilde{\gamma}(t) = \gamma_n(e^t)$, $\widetilde{\theta}(t) = \theta_n(e^t)$. The fundamental system of solutions for (13) is:

$$\left\{e^{-t} \quad , \quad e^{2t}\right\}, \tag{14}$$

so the general solution is:

$$\widetilde{\gamma}(t) = c_1(t)e^{-t} + c_2(t)e^{2t}, \tag{15}$$

where the functions $c_1(t)$ and $c_2(t)$ are determined from the following conditions:

$$\begin{cases} c_1'(t)\cdot e^{-t} + c_2'(t)\cdot e^{2t} = 0 \\ -c_1'(t)\cdot e^{-t} + 2c_2'(t)\cdot e^{2t} = -\theta_n^n(t)e^{4t} \end{cases} \tag{16}$$

Solving (16) and substituting in (15) we find that the general solution is:

$$\widetilde{\gamma}_n(t) = -\frac{1}{3}e^{2t}\int e^{2t}\widetilde{\theta}_n^n(t)dt + \frac{1}{3}e^{-t}\int e^{5t}\widetilde{\theta}_n^n(t)dt + K_1e^{2t} + K_2e^{-t} \tag{17}$$

where the constants K_1 and K_2 are determined using the boundary conditions (12).

Going back from the variable t to ξ we find the following expression for $\gamma_n(\xi)$:

$$\gamma_n(\xi) = -\frac{1}{3}\xi^2\int \xi\theta_n^n(\xi)d\xi + \frac{1}{3}\frac{1}{\xi}\int \xi^4\theta_n^n(\xi)d\xi + K_1\xi^2 + \frac{K_2}{\xi}. \tag{18}$$

which enables us to find the constants K_1 and K_2 using (12):

$$K_1 = \left[\frac{1}{3}\int \xi\theta_n^n(\xi)d\xi\right]\Bigg|_{\xi=\xi_1} ; \quad K_2 = -\left[2\int \xi^4\theta_n^n(\xi)d\xi\right]\Bigg|_{\xi=0}. \tag{19}$$

We note that in (18) and (19) is involved the Lane-Emden function of index n. But as we know, there are only three cases ($n \in \{0, 1, 5\}$) whose exact form could be written (see Chandrasekhar, 1939). Further we will use an approximate form of it to be able to compare our results with the numerical results of Monaghan (1965).

4. CERTAIN SOLUTION FOR EQUATION (10)

Let us substitute in (18) $n = 0, 1$ to get exact solutions for (10). For $n = 0$ we get the solution found by Ferraro (1954), and for $n = 1$ the Monaghan's solution (1965):

for $n = 0$

$$\theta_0 = 1 - \frac{1}{6}\xi^2\gamma_n(\xi) = \xi^2 - \frac{\xi^4}{10}$$

for $n = 1$

$$\theta_1 = \frac{\sin\xi}{\xi}\gamma_n(\xi) = \xi\sin\xi + 2\cos\xi - \frac{2\sin\xi}{\xi} + \frac{1}{3}\xi^2$$

For other values of n we know only approximate or numerical solutions for (11). In the sequel we are going to use the first order Padé approximation in $\xi = 0$ for the Lane–Emden function[1] (Pascual, 1977), which means:

$$\theta_n^{[1,1]}(\xi) = \frac{60 + (3n - 10)\xi^2}{60 + 3n\xi^2}. \tag{20}$$

Choosing $n = 1$ in (18) and (22) we find the following expression for $\gamma_1(\xi)$

$$\gamma_1(\xi) = \frac{1}{3}\xi^2\left(138.3339469 + \frac{7}{6}\xi^2 - \frac{100}{3}\ln(60 + 3\xi^2)\right) \tag{21}$$

$$+\frac{1}{3}\,\frac{-\frac{7}{15}\xi^5 + \frac{200}{9}\xi^3 - \frac{4000}{3}\xi + \frac{8000}{3}\sqrt{5}\mathrm{arctan}\left(\frac{1}{10}\xi\sqrt{5}\right)}{\xi} \tag{22}$$

For $n = 2$, we get $\gamma_2(\xi)$

$$\gamma_2(\xi) = \frac{1}{3}\xi^2\left(-37.98177011 - \frac{2}{9}\xi^2 + \frac{1250}{9}\frac{1}{10 + \xi^2} + \frac{100}{9}\ln(10 + \xi^2)\right) + \frac{1}{3} \tag{23}$$

$$\times\left(\frac{4}{45}\xi^5 - \frac{200}{27}\xi^3 + 500\xi + \frac{12500}{9}\frac{\xi}{10 + \xi^2} - \frac{5750}{9}\sqrt{10}\mathrm{arctan}\left(\frac{1}{10}\xi\sqrt{10}\right)\right)/\xi \tag{24}$$

For $n = 3$ we obtain $\gamma_3(\xi)$

$$\gamma_3(\xi) = \frac{1}{3}\xi^2\left(.7999286730 + \frac{1}{1458}\xi^2 + \frac{2000000}{2187}\frac{1}{(20 + 3\xi^2)^2} - \frac{20000}{729}\frac{1}{20 + 3\xi^2}\right. \tag{25}$$

[1] In the neighbourhood of $\xi = 0$, the power series expansion of the Lane–Emden function has only even terms. So, we have to approximate a power series on ξ^2. So, in fact, here θ_n is rather a function of ξ^2 than a function of ξ. This the reason why $\theta_n^{[1,1]}$ is termed "first order Padé approximant" it being really of first order with respect to ξ^2 (not with respect to ξ).

$$-\frac{100}{729}\ln(20+3\xi^2)\Big)+\frac{1}{3}\left(-\frac{1}{3645}\xi^5+\frac{200}{2187}\xi^3-\frac{44000}{2187}\xi\right. \tag{26}$$

$$+\frac{40000000}{6561}\frac{\xi}{(20+3\xi^2)^2}-\frac{6200000}{6561}\frac{\xi}{20+3\xi^2} \tag{27}$$

$$\left.+\frac{76000}{2187}\sqrt{15}\arctan\left(\frac{1}{10}\xi\sqrt{15}\right)\right)/\xi \tag{28}$$

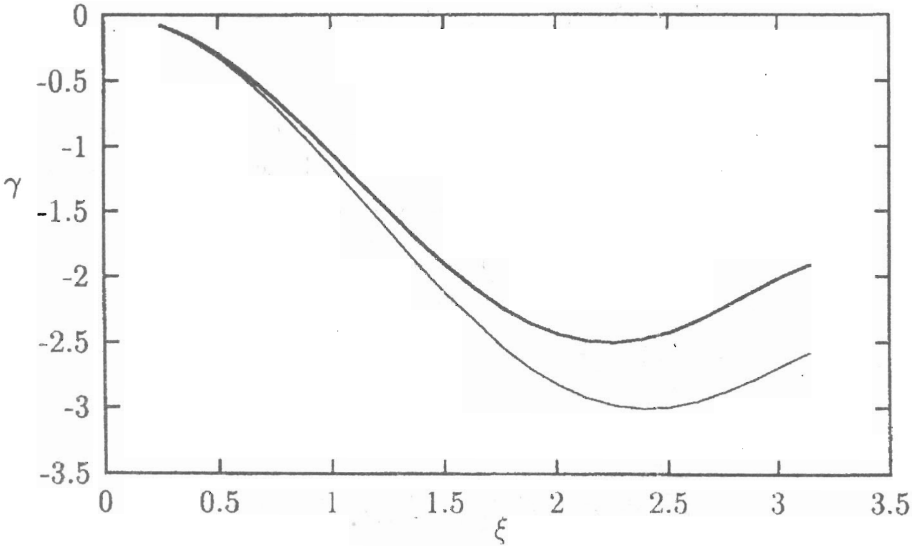

Fig. 1. — The poloidal magnetic field for $n = 1$.

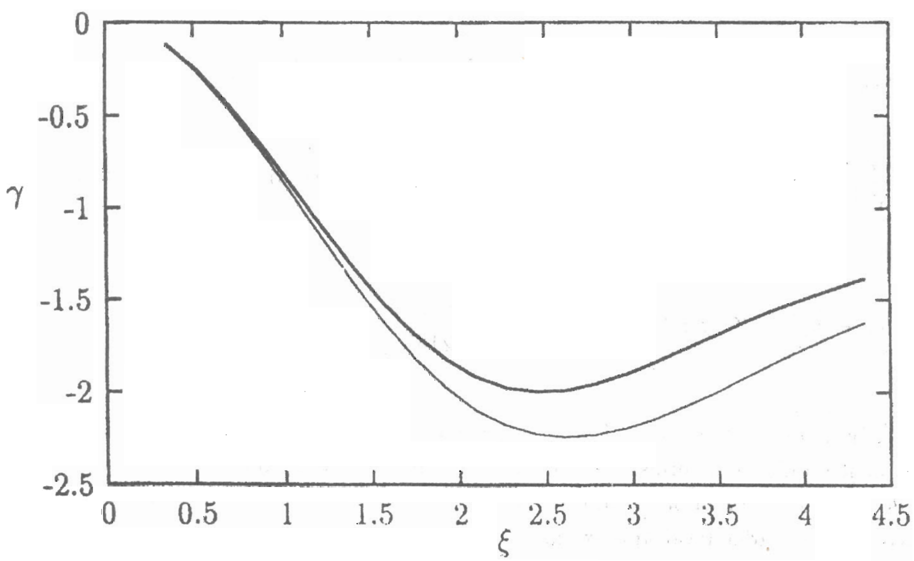

Fig. 2. — The poloidal magnetic field for $n = 2$.

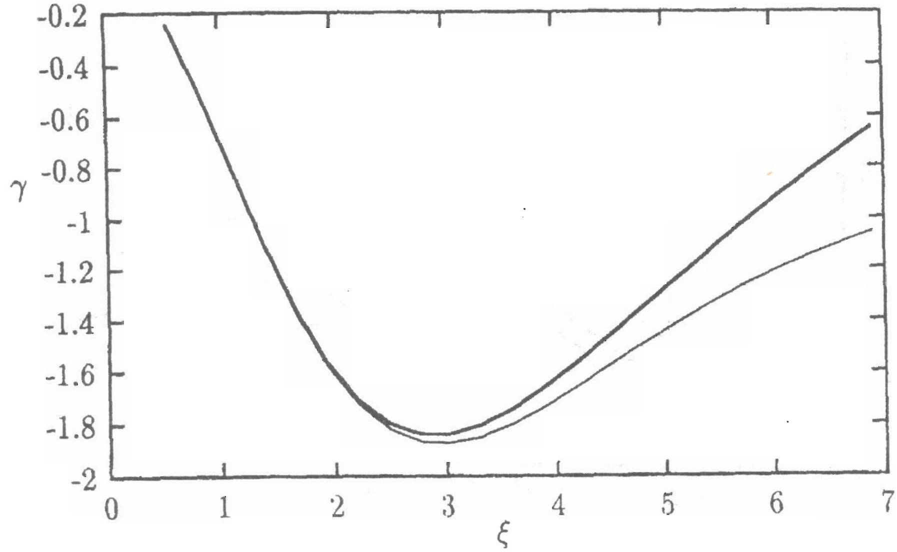

Fig. 3. — The poloidal magnetic field for $n = 3$.

In figures 1, 2 and 3 we have represented (23), (24) and (25) (thin line) and also the numerical solution (Monaghan, 1965)[2] (thick line). Our solution approximates the numerical one with the mean error 0.3, 0.17 and 0.11 respectively. At the surface of the star our solution is far from the numerical one, mainly, because we used a first order Padé approximant in $\xi = 0$ for θ_n. To overcome this fact we could use a higher order Padé approximation or to part the star in envelopes in which we compute different Padé approximants. But these approaches involved more calculations and complicated the analytical form of the solution.

In figure 4 we analyze the magnetic field at different depths in the star (on x-axis we represent ξ/ξ_1, where ξ_1 is the first zero of the Lane–Emden function) and we see that increasing the polytropic index, the main value of the field is found deeper in the star. The maximum value of the field does not depend strongly on the polytropic index for stars with strong central condensation, as we can see in figure 4.

[2] The functions $\gamma_n(\xi)$ and $B_0(\xi)$ from Monaghan's article – dimensionless functions – are in the ratio -2. So, in fact, we represent in our pictures $-2\gamma_n(\xi)$ and $B_0(\xi)$.

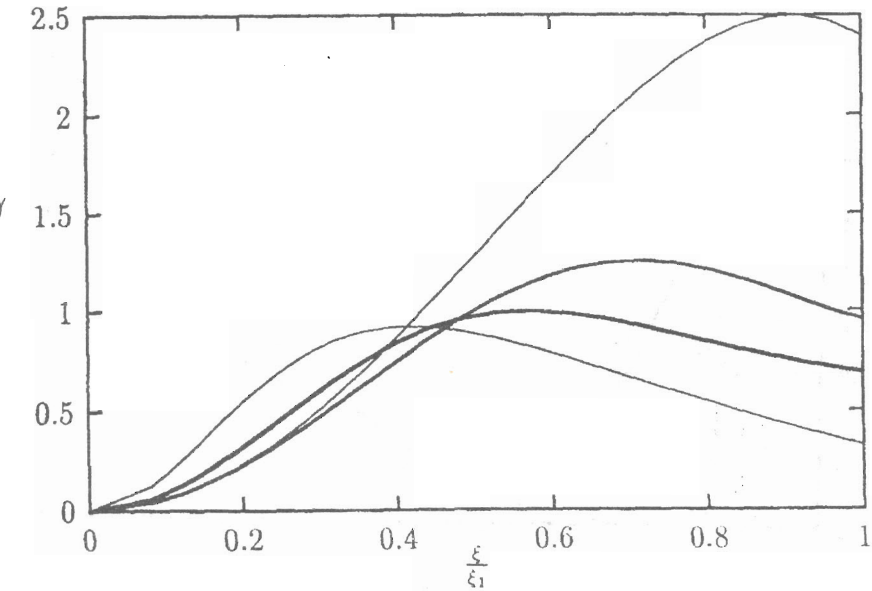

Fig. 4. — The magnetic field versus the star depth for $n \in \{0, 1, 2, 3\}$; the first line from the top is for $n = 0$.

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