AN INVERSE PROBLEM FOR ISOLATED ORBITS

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Abstract. We examine the question of finding all potentials $V(x, y)$ which can produce a specific *isolated* planar orbit $f(x, y) = 0$, given in advance and traced by a material point for adequate initial conditions. We study in some detail an example to clarify the difference of this version of the problem from the usual version dealing with a *monoparametric family* of orbits $f(x, y) = constant$.

Key words: celestial mechanics — inverse problem of dynamics.

1. INTRODUCTION

We shall refer to the version of the inverse problem of dynamics which seeks the force fields or, most frequently, the potentials $V(x, y)$ which can given rise, for adequate initial conditions, to certain orbits given in advance and traced in the Cartesian xy-plane by a material point of units mass. Regarding the multitude of the given orbits we shall refer either to a *monoparametric family of curves* described by the equation

$$f(x, y) = c \tag{1}$$

or to a *single curve*

$$F(x, y) = 0. \tag{2}$$

Whittaker (1944), for instance, presents Dainelli's work for the family (1) and offers the force components generating all orbits (1). He also states clearly that "on varying the constant c, this equation will represent a family of curves". But Dainelli himself in his 1880 report refers to the single orbit (2).

Working in an inertial frame, Szebehely (1974) derived his partial differential equations in $V(x, y)$ for a given family of curves (1) but his derivation holds for *isoenergetic* families, i.e. having E = constant for all members of (1). Few years later, Morrison (1977) generalized this equation to account for energy dependence

$$E = E(c) \tag{3}$$

given in advance along the family.

Besides, from the old days up to now, one can locate in the literature references to the inverse problem for *one single orbit* (2), given in advance. We quote, for instance, Suslov (1890), Rajaraman (1979), P. du T. der Merwe (1991), Antonov and Timoshkova (1993), who formulate the inverse problem with equation (2). In the framework of the restricted three-body problem Drâmbă also used equation (2) in a rotating frame. We mention also Kasner's (1909) work on the corresponding *direct problem* and the third order ordinary differential equation in $y(x)$ for "the definite trajectory" $y = y(x)$ with coefficients depending on the components.

So, in some cases, the presence of the varying parameter c in the right hand side of equation (1) was simply ignored or taken as equivalent to its absence. But, at certain instances (*e.g.* Szebehely et al. (1980)) the role of c was misinterpreted and, in the pertinent formulae, instead of equation (1), the equation $F(x, y, c) = 0$ was used. As a result, one ended up with potentials depending on c, which, of course, cannot be.

From the viewpoint of an astronomer, having *one* specific observed orbit (perhaps, *some* orbits) is the usual situation, whereas it is practically impossible to possess a *continuous* set of such observed orbits.

The aim of this paper is to clarify the role of the parameter c and comment on the essential difference between the two inverse problems dealing with *one isolated orbit* (2) and a *family of orbits* (1). It will become clear that the solution of the *one-orbit problem* is conveyed to the known framework where the pertinent partial differential equations hold true.

2. THE TWO BASIC EQUATIONS

All potentials $V(x, y)$ which can generate the monoparametric family (1) traced with a preassigned energy dependence (3) are given by Szebehely's first order partial differential equation

$$V_x + \gamma V_y + \frac{2\Gamma}{1+\gamma^2}(E - V) = 0 \tag{4}$$

where

$$\gamma = \frac{f_y}{f_x}, \quad \Gamma = \gamma\gamma_x - \gamma_y. \tag{5}$$

It is easily seen from the above equations that

(i) to each function $f(x, y)$ there corresponds *one* function $\gamma(x, y)$ and, vice versa, to each $\gamma(x, y)$ there corresponds *one* family (1). So, assigning the function $\gamma(x, y)$ is equivalent to specifying the family (1) and, since, E being given, the function $f(x, y)$ no longer appears in equation (4), we can refer to the "family of orbits $\gamma(x, y)$".

(ii) in general, as many potentials as an arbitrary function of a certain argument would allow can produce, for adequate initial conditions, the given family $\gamma(x, y)$. Each of these potentials, of course, produces, for random initial conditions, other orbits not included in (1).

On the other hand, not all the given geometrical information (1) becomes mechanical reality too. For actual motion to take place, of course, one needs to make sure that $E \geq V(x, y)$ as the massive point is moving and this requirement leads to the inequality (Bozis and Ichtiaroglou, 1994)

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0. \tag{6}$$

The so-called *family boundary curves* define *regions of the xy-plane* where motion along the various members of (1) is allowed. Only these curves (1) or those parts of the curves (1) which lie inside the allowed region (6) are then admitted as *actual orbits*. Since the (constant along *each* orbit) energy E varies from orbit to orbit, the above curves do not, generally, coincide with the well-known *zero velocity curves* which refer to orbits of the same energy.

The second equation, relating only potentials and families of orbits (1), is (Bozis, 1984)

$$-V_{xx} + kV_{xy} + V_{yy} = \lambda V_x + \mu V_y \tag{7}$$

with

$$k = \frac{1-\gamma^2}{\gamma}, \quad \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \quad \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{8}$$

Equation (7) is linear of the second order and its general solution introduces *two* arbitrary functions. This is in agreement with the multitude of solutions of equation (4) which is solved, each time, for an arbitrary selection of the function $E = E(c)$.

Usually we make equations (7) and (4) cooperate as follows: if a pair of orbits $\gamma(x, y)$ and potentials $V(x, y)$ is compatible (i.e. if it satisfies equation (7)), the corresponding total energy E can be found from equation (4). The energy must be of the form (3) and this constitutes a criterion of the correctness of the calculations involved.

3. POTENTIALS GENERATING ISOLATED TRAJECTORIES

Let us now face an inverse problem on the basis of *one isolated trajectory* (2) given in advance. We assume that the equation $F(x, y) = 0$ stands for *one* smooth (closed or not) geometrical curve and we put the question: Which potentials are compatible with this specific trajectory?

At this stage we do not care as to whether actual motion takes place on the entire curve (2) or is a libration or an asymptotic motion on a certain arc of (2).

First of all, let us make clear that our equations (4) and (7) hold true for families of orbits which are *at least* monoparametric, expressed by equation (1).

Comment: For autonomous systems, the given family (1), may be *at most* three-parametric.Two and three-parametric families $F(x, y, b) = c$ and $F(x, y, a, b) = c$ were studied by Bozis (1983) and by Xanthopoulos and Bozis (1983) respectively. It was shown that, in general, no potential exists which can give rise to such families given in advance arbitrarily. If, however, the given functions $F(x, y, b)$ or $F(x, y, a, b)$ satisfy certain conditions, then there exists a potential which can be determined "almost" uniquely.

So or otherwise the solutions of equations (4) and (7) are defined and the equations themselves are meaningful in open domains of the *xy*-plane and by no means on certain curves (2) of the plane. Clearly, all members of (1), for values of c taken from an open interval of R, lie inside such a domain.

So then, to answer the above question we are forced to classify the given trajectory (2) to a certain family (1), compute from (5) the corresponding function $\gamma(x, y)$ and solve for $V(x, y)$ equation (7). Every solution of (7) will also generate the given trajectory (2) which belongs to the family.

In classifying (2) one first feels tempted to consider that the given curve $F(x, y) = 0$ is simply the member of the family $F(x, y) = c$ for $c = 0$. The problem then is transferred to the problem of solving equation (7) with $\gamma = F_y / F_x$. This is of course correct and, in fact, people mentioned in the Introduction treat $F(x, y) = 0$ as if it were the same with $F(x, y) = c$. Yet it is only one way, out of many possible ways, of answering our question because the single curve $F(x, y) = 0$ can be classified as a member of many other monoparametric families (1), having different functions $\gamma(x, y)$ and, consequently, leading to different differential equations (7). These equations are not expected to have the same solutions, although some solutions may be common. Therefore, new admissible potentials (in the sense that they create the isolated orbit with equation $F(x, y) = 0$) will be added, due to the new selection of the monoparametric family to which $F(x, y) = 0$ belongs.

To clarify and support our argument we propose to work out in some detail a simple, yet typical example.

4. EXAMPLE

The isolated curve

$$x^2 + y^2 = 1 \tag{9}$$

belongs to the monoparametric family of concentric circles

$$x^2 + y^2 = c_1 \tag{9a}$$

for $c_1 = 1$. However, the same curve (9) may be taken, *e.g.* as a member of the monoparametric family of conic sections

$$\frac{1 - x^2}{y^2} = c_2 \tag{9b}$$

for $c_2 = 1$, not to mention other possible classifications as, for instance, the family $\dfrac{-y^2 + \sqrt{y^4 + 4x^2}}{x^2} = c_3$ for $c_3 = 2$, etc.

In what follows we shall deal with the two families (9a, b). The functions $\gamma(x, y)$ are, respectively: $\gamma_1 = y/x$. and $\gamma_2 = (1 - x^2)/xy$. Calculating the coefficients k, λ, μ, from equations (8) and inserting into equation (7) we obtain, respectively,

$$-V_{xx} + \frac{x^2 - y^2}{xy} V_{xy} + V_{yy} - \frac{3}{x} V_x + \frac{3}{y} V_y = 0 \tag{10a}$$

and

$$-V_{xx} + \frac{(x^2 - 1)^2 - x^2y^2}{xy(x^2 - 1)} V_{xy} + V_{yy} - \frac{3}{x} V_x + \frac{3}{y} V_y = 0. \tag{10b}$$

The general solution of equation (10a) is known from Broucke and Lass (1977) and Molnár (1981). In polar coordinates, r, θ it reads

$$V(r, \theta) = g(r) + \frac{1}{r^2} h(\theta) \tag{11}$$

where g and h are arbitrary functions of their respective arguments.

The second equation (10b) is, obviously, different from (10a). Actually, it so happens that $k_2 \neq k_1$, whereas $\lambda_2 = \lambda_1$, $\mu_2 = \mu_1$. One then understands that all *sepa-*

rable (in the coordinates x, y) solutions $V(x, y)$ of equation (10a), i.e. all solutions with $V_{xy} = 0$, are also solutions of (10b). Being easily detected, these common solutions are, apart from an additive constant,

$$V_{sep} = \frac{a_1}{x^2} + \frac{a_2}{y^2} + a_3(x^2 + y^2) \tag{12}$$

where a_1, a_2, a_3 are constants.

The set of potentials (12) is a subset of (11) corresponding to $g(r) = a_3 r^2$ and $h(\theta) = \frac{a_1}{\cos^2\theta} + \frac{a_2}{\sin^2\theta}$. From equation (4) it can be found that the circles (9a) are traced by the potential (12) with energy

$$E_1 = 2a_3c_1 \tag{13a}$$

whereas the conics (9b) are traced by the same potential (12) with

$$E_2 = a_3 + a_1 + \frac{a_3 - a_1}{c_2} + a_2c_2(1 - c_2). \tag{13b}$$

Notice that the unit circle (9) is traced with $E = 2a_3$.

We did not manage to find the general solution of the second order equation (10b). However, we did find a set of solutions of (10b) whose multiplicity is introduced through one arbitrary function (not two). What we did was to solve Szebehely's first order equation (4) for the family (9b) under the additional assumption that all members of this family are traced with energy $E = 0$. For this case equation (4) reads

$$V_x + \frac{1 - x^2}{xy} V_y = \frac{2(x^2 - 1)}{x[x^2y^2 + (1 - x^2)^2]} V. \tag{14}$$

Forming and solving the corresponding system of subsidiary equations we found the general solution of (14)

$$V_{isoen} = \left[2 - \left(x^2 + y^2 + \frac{1}{x^2}\right)\right] H[2\ln|x| - (x^2 + y^2)] \tag{15}$$

where H is an arbitrary function of its argument.

As expected, all potentials (15) are also solutions of equation (7), as applied for the family (9b). In general, these solutions are not separable, i.e. $V_{xy} \neq 0$. Each of these potentials can create (9b) as a family of *isoenergetic orbits*, with $E = 0$. On the other hand, we have shown that each potential (12) creates the family (9b) with energy E, given by (13b). Then, due to the linearity in $V(x, y)$ of the partial differential equation (7), the two solutions can be added to obtain a "richer" solution

$$V = V_{sep} + V_{isoen} \tag{16}$$

producing orbits (9b) with total energy $E = 0 + E_2 = E_2$, given by (13b).

In general, a potential (16) does not create the entire family (9a) and, also, a potential (11) does not create the entire family (9b). However, all potentials (16) and (11) can produce the single orbit (9) which belongs to both families (9a) and (9b).

Needless to say that, in addition to the above, one could find other potentials giving rise to the specific orbit (9). To this end one should classify the orbit (9) to other monoparametric families and attempt to solve the corresponding equation (7). New solutions would then appear.

Comment: The expression (16) does not even include all potentials producing (9) considered as a member of the family (9b). Other solutions would spring if, in solving equation (4) as we did, instead of $E = 0$, we had selected another energy dependence $E_2 = E_2(c_2)$, not given by (13b) for a certain triplet a_1, a_2, a_3. But no one could guarantee that the corresponding equation (4) would actually be solvable, as it happened in the zero energy case.

5. ARCS OF ALLOWED MOTION

Our main concern in sections 3 and 4 has been to find all potentials which are compatible with the isolated trajectory (9). To this end, having classified (9) in various families (9a), (9b) etc., we tried to solve the corresponding differential equations (7).

But does, actually, the entire curve (9) stand for an *orbit* of a massive point? And is the motion a rotation on the entire circle $x^2 + y^2 = 1$ or a libration or an asymptotic motion on a certain arc of it?

The answer to these questions, coming from inequality (6), goes as follows: It all depends on the potential and the family to which (9) has been clasified. Only this part of the curve $x^2 + y^2 = 1$ (if any) which happens to lie inside allowed region defined by (6) will be admitted as a *real orbit*.

Take, for instance, the potential

$$V = \left[2 - \left(x^2 + y^2 + \frac{1}{x^2}\right)\right][2 - (x^2 + y^2)2\ln|x|] \tag{17}$$

of the form (15). For $c_2 > 0$, among the ellipses (9b), all traced with zero energy, the circle (9) is included for $c_2 = 1$. Since $2 - (x^2 + y^2 + 1/x^2) \leq 0$ everywhere in the plane, the requirement $0 \geq V(x, y)$ or, equivalently, because all motions are isoenergetic, inequality (6) leads to

$$x^2 + y^2 - 2 \ln | x | \leq 2. \tag{18}$$

The two (symmetrical with respect to both axes Ox, Oy) shaded domains in Fig. 1 are defined by (18) and represent the allowed regions of motion for members of (9b) traced by (17). On the other hand, as can be checked easily, the points $A_0(1, 0)$ and $A_0'(-1, 0)$ of Fig. 1 (which belong to all conics $x^2 + c_2 y^2 = 1$) are *unstable equilibrium points of the potential* (17). Since, except for $V_x = V_y = 0$ at A_0 (and A_0'), it is also $V(A_0) = 0$ (and $V(A_0') = 0$), one understands that, as regards *e.g.* the unit circle (9), real motion on it with zero energy will be *asymptotic*, tending to A_0 (or A_0') from $y > 0$ or from $y < 0$, depending on the initial position.

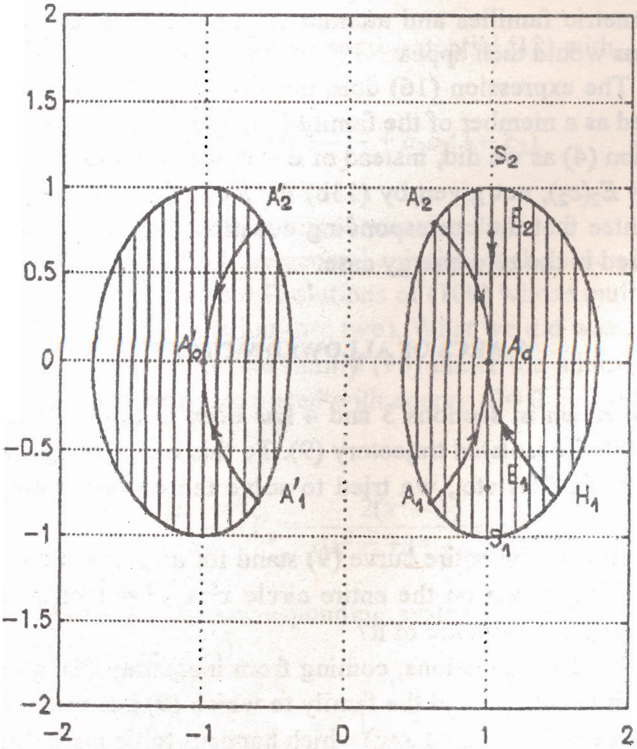

Fig. 1. — Conic sections (9b), generated by the potential (17) represent real orbits (arcs of asymptotic motion) only inside the shaded regions. In particular the asymptotic motion on the arc A_2A_0 of the unit circle is shown by a heavy line. The line H_1A_0 with H_1 (1.47951, −0.77102) corresponds to asymptotic motion on the hyperbola (9b) with $c_2 = -2$.

To verify this result we integrated numerically the system

$$\ddot{x} = -V_x, \quad \ddot{y} = -V_y \tag{19}$$

with V given by (17) and with initial conditions

$$x_0 = 0.6065306597, \quad y_0 = 0.7950600976, \quad \dot{x}_0 = 0, \ \dot{y}_0 = 0.$$

Starting with zero velocity at the point A_2 of Fig. 1, the unit mass moved indeed very accurately on the circular arc A_2A_0 tending "asymptotically" to A_0. In fact it reached the closed vicinity of A_0 in finite time with velocity very close to zero, it stayed there for a time longer than it took it to trace the entire arc A_2A_0 and then, due to the instability of the equilibrium point A_0, it went away from A_0. The energy remained constant but the unit mass no longer travelled along the unit circular arc. It remained, however, inside the right shaded region of Fig. 1 because the total energy is zero and inequality (18) gives not only the family boundary curve (6) but the zero velocity curve $0 \geq V(x, y)$, as well.

Since the point $A_0(1, 0)$ belongs to all conics (9b), the motion is asymptotic not only on the arc A_2A_0 but on all arcs (9b), terminating in A_0, as *e.g.* on the arc H_1A_0 of the hyperbola $x^2 - 2y^2 = 1$, as well as on the straight line segment S_2A_0 which corresponds to $c_2 > 0$ and which separates the elliptical ($c_2 > 0$) from the hyperbolic ($c_2 < 0$) arcs of the family (9b).

The rectilinear motions on S_2A_0 can be found analytically: Indeed, for $x = 1$, the first equation (19), applied for the potential (17), becomes an identity whereas the second equation (19) gives

$$\ddot{y} = 2y(1 - 2y^2). \tag{20}$$

For the zero energy asymptotic motion on S_2A_0, starting with zero velocity at the point $S_2(1, 1)$, the solution of (20) is

$$y = \frac{2e^{-\sqrt{2}t}}{1 + e^{-2\sqrt{2}t}}, \tag{21}$$

giving, for $t \to \infty$, $y \to 0$.

The motion on S_1A_0 (Fig. 1) is symmetrical to (21) and, of course, analogous results can be found for $x < 0$.

The question now arises: are there potentials creating *librational* motion on the arc A_1A_2 of Fig. 1? The answer is in the affirmative. Actually, there are several such potentials. As an example we give the potential (Bozis and Ichtiaroglou, 1994)

$$V(r, \theta) = \frac{1}{r} - \frac{a}{2r^2}(1 + \cos\theta) \tag{22}$$

of the form (11) creating arcs of circles (9a) *inside* the cardioid

$$r = a(1 + \cos\theta). \tag{23}$$

We only need adjust the value of $a = (1 + e^{-1/2})^{-1}$ so that (22) intersects the boundary of (18) at the same points $A_1(e^{-1/2}, -(1 - e^{-1})^{1/2})$ and $A_2(e^{-1/2}, (1 - e^{-1})^{1/2})$.

Comments: 1. The points $E_1(1, -\sqrt{2}/2)$, $E_2(1, \sqrt{2}/2)$ are also unstable equilibrium points of the *two-dimensional potential* (17) but they correspond to total energy $-1/4$. So, they play no role in zero energy motions created by (17). Actually E_2 is reached by the mass moving according to (21) in time $t = \dfrac{\ln(\sqrt{2}+1)}{\sqrt{2}}$ with velocity $v = -\dfrac{\sqrt{2}}{2}$.

2. A detailed consideration reveals that as regards *rectilinear motions* on the line $x = 1$, the point A_0 is unstable, whereas the points E_1, E_2 are stable.

3. The energy of the asymptotic motions on A_1A_0 or A_2A_0 due to the potential (17) is zero, whereas the energy of the librational motion due to (22) is 1/2.

4. The two potentials (17) and (22) have in common the arc A_1A_2 but not $A_1'A_2'$ on which real motion is not allowed by (22).

5. Potentials can be found allowing for rotational motion on the entire circle $x^2 + y^2 = 1$. Such is, *e.g.*, the Newtonian potential $V = -1/r$.

REFERENCES

Antonov, V. A. and Timoshkova, E. I., 1993, *Astron. Rep.*, **37** (2), 138.
Bozis, G.: 1983, *Celest. Mech.*, **31**, 129.
Bozis, G. and Ichtiaroglou, S.: 1994, *Celest. Mech.*, **58**, 371.
Broucke, R. and Lass, H.: 1977, *Celest. Mech.*, **16**, 215.
Dainelli, U.: 1880, Giornale di Mat. di Battaglini, **XVIII**, 271.
Drâmbă, C.: 1963, *St. Cerc. Astron.*, **8**, 7.
Kasner: 1909, *Differential Geometric Aspects of Dynamics,* Princeton Colloquium, 1909, New York, A.M.S. 1913 (reprinted 1933).
Molnár, S.: 1981, *Celest. Mech.*, **29**, 81.
Morrison, F.: 1977, *Celest. Mech.*, **16**, 39.
P. du T. van der Merve: 1991, *Physics Letters A*, **156**, 216.
Rajaraman, R.: 1979, *Phys. Rev. Let.*, **42**, 200.
Suslov, G. K.: 1890, *On a force function admitting given integrals,* Kiev (in Russian).
Szebehely, V.: 1974, in E. Proverbio (ed) *Proceedings of the Int. Meeting on Earth's Rotation by Satellite Observations*, Univ. of Cagliari, Bologna, Italy.
Szebehely, V., Lundberg, J. and McGahee, W.: 1980, *Astrophys. J.* **239**, 880.
Whittaker, E. T.: 1944, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* Dover Publ., New York.
Xanthopoulos, B. and Bozis, G.: 1983, in *Dynamical Trapping and Evolution in the Solar System,* I.A.U. Coll. 74, (eds. Y. Kozai and V. Markellos), Reidel, Dordrecht, 353.

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