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GRAVITATIONAL COLLAPSE OF BOSE-EINSTEIN CONDENSATE DARK MATTER HALOS WITH LOGARITHMIC NONLINEARITY

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Abstract. If dark matter is composed of massive bosons, a Bose-Einstein Condensation process must have occurred during the cosmological evolution. Therefore galactic dark matter may be in a form of a condensate, characterized by a strong self-interaction. One of the interesting forms of the self-interaction potential of the condensate dark matter is the logarithmic form. In the present work we investigate one of the astrophysical implications of the condensate dark matter with logarithmic self-interaction, namely, its gravitational collapse. To describe the condensate dark matter we use the Gross-Pitaevskii equation, and the Thomas-Fermi approximation. By using the hydrodynamic representation of the Gross-Pitaevskii equation we obtain the equation of state of the condensate, which has the form of the ideal gas equation of state, with the pressure proportional to the dark matter density. In the Thomas-Fermi approximation, the evolution equations of the condensate reduce to the classical continuity, and Euler equations of fluid dynamics. We obtain the equations of motion of the condensate radius in spherical symmetry, by assuming certain particular forms for the velocity and density of the condensate. The collapse time required for the formation of a stable macroscopic astrophysical object is obtained in an integral form, and explicit numerical estimations for the formation of astrophysical objects with masses ranging from $10^6 M_\odot$ to $10^{12} M_\odot$ are presented.

Key words: Bose-Einstein condensates, Gross-Pitaevskii equation, hydrodynamical representation, collapsing condensate, logarithmic nonlinearity.

1. INTRODUCTION

The discovery of the ultra-luminous quasar SDSS J010013.02+280225.8 at redshift $z = 6.30$, with a central black hole mass of order $M = 1.2 \cdot 10^{10} M_\odot$ (Wu *et al.*, 2015)* has led to fundamental theoretical questions on the formation and growth of supermassive black holes (SMBH) during astrophysically short time intervals in the early Universe.

Up to now, more than 200 quasars have been discovered at redshift $z > 6$, with X-ray emission from quasars being detected up to redshift $z = 7.5$ (Wang *et al.*, 2021a). As of today, the most distant quasar known is J031343.84–180636.4, with a redshift of $z = 7.642$ and a mass of $M = 1.6 \cdot 10^9 M_\odot$. The existence of a SMBH just ~ 670 million years after the Big Bang places severe limits on the current standard paradigms of stellar formation (Wang *et al.*, 2021b). ULAS J1342+0928, the second-most distant known quasar detected up to now, is located at a redshift of $z = 7.54$, and when it emitted the light detected today, its age was 690 million years after the Big Bang (Banados *et al.*, 2018).

Evidence for the existence of a material that is more abundant than baryonic matter, constituting 27% of the universe, was first found in 1933 by the astronomer Fritz Zwicky (1933), who discovered discrepancies between the observational and theoretical data. The observed mass was only a small part of the mass required to prevent the galaxies from escaping the cluster Coma. Many years after this problem was acknowledged, dark matter became a paradigm of different types of astronomical measurements that defied expectations regarding the fabric of space. As of 2022, no experiment confirmed the existence of dark matter particles, and the galactic rotational curves issue is still under research. Dark energy poses a similar problem to scientists, as its effects are clearly present but its underlying nature is yet to be determined. As the evolution on cosmological scales is determined by the initial conditions, the interactions between galaxies and the intergalactic medium, many scenarios of structure formation are ruled by the cold dark matter paradigm.

An interesting theoretical, and observational possibility is that dark matter can exist in the form of a Bose-Einstein Condensate. Such a possibility was initially considered by (Membrado and Aguerri, 1996; Membrado, 1998), and later on by Sin (1994). When the temperature of a bosonic system drops below the critical temperature,

$$T_C = \frac{2\pi\hbar}{mk_B}\left[\frac{n}{\zeta(3/2)}\right]^{2/3}, \tag{1}$$

where n is particle number density, $\hbar$ is the Planck constant, k_B is Boltzmann's constant, while $\zeta(3/2)$ denotes the Riemann zeta function, a phase transition occurs in the system, with most of the bosons occupying the lowest quantum state (Dalfovo *et al.*, 1999; Strigari, 2013). The existence of bosonic particles, such as fundamental scalars, allows the possibility of a self-gravitating BEC on an astrophysical scale.

The process of Bose-Einstein condensation allows bosons to occupy the lowest

*$M_\odot = 2 \cdot 10^{30}$ kg is the Solar mass

possible quantum state simultaneously, in contrast to fermions with states restricted by the Pauli exclusion principle. This phenomenon is well modeled by the nonlinear Schródinger equation (SE) for the macroscopic wave function, also known as the Gross-Pitaevskii equation (GP). The Gross-Pitaevskii equation is a long-wavelength approach widely used to describe dilute Bose-Einstein Condensates (BECs). The Gross-Pitaevskii equation has the following general form (Dalfovo *et al.*, 1999; Strigari, 2013),

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+\Psi\int|\Psi\left(x',t\right)|^2V\left(|x-x'|\right)dx', \tag{2}$$

where m is the mass of a particle in the BEC, and $V\left(|x-x'|\right)$ is the interaction potential between bosons. For a weakly interacting system, the potential is simplified to $V\left(|x-x'|\right)=V_0\delta\left(|x-x'|\right)$, giving the Gross-Pitaevskii equation with quadratic nonlinearity,

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+V_0|\Psi|^2\Psi. \tag{3}$$

For bosonic systems confined by the gravitational potential, V must satisfy the Poisson equation, given by (Boehmer and Harko, 2007),

$$\Delta V=4\pi Gm\rho, \tag{4}$$

where ρ is the density of dark matter. The coupled system of equations (3) and (4) allow a full description of the properties of the dark matter halos, and of the behavior of the galactic rotation curves (Boehmer and Harko, 2007; Craciun and Harko, 2020). In particular, the density distribution of the dark matter condensate can be obtained as

$$\rho(r)=\rho_c\frac{\sin(kr)}{kr}, \tag{5}$$

where ρ_c is the central density, while k is a constant. Bose-Einstein Condensate systems with quadratic nonlinearity have been extensively studied in the literature, and they have many applications in both cosmology and astrophysics. In particular, it was pointed out in Harko (2011b) that Bose-Einstein Condensate dark matter can solve the core-cusp problem of the galactic astrophysics.

Assuming a flat Friedmann-Robertson-Walker geometry and a nonlinear term related to the density of the wave function, the cosmological evolution of the finite temperature condensed dark matter was investigated in Harko and Mocanu (2012). The cosmological perturbations of the Bose-Einstein Condensate dark matter was considered in Harko (2011a). Another relevant example is a Bose-Einstein Condensate (BEC) driven by the time-dependent Schrödinger-Poisson (SP) system of equations (Gonzalez and Guzman, 2011). These equations describe a condensate of scalar field particles at zero temperature in the mean field approximation. In cylindrical coordinates they have the following form,

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{\partial^2\Psi}{\partial z^2}\right) + U\Psi + \Lambda|\Psi|^2\Psi, \tag{6}$$

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = \Psi^*\Psi \tag{7}$$

where Ψ and U are the wave function and the gravitational potential, respectively.

Bose-Einstein Condensates are macroscopically sized quantum objects, making them an excellent candidate for testing fundamental physics, including the Schrödinger equation and General Relativity, as well as possible intersections between these theories. Quantum reflection of up to 60% has been seen in BEC experiments (Pasquini *et al.*, 2006). The discussion of freely evolving degenerate quantum gases at large evolution times achieved in microgravity experiments, enabled the possibility of future testing the nonlinear form of Schrödinger's equation (Herrmann *et al.*, 2010).

Studying the effects of interaction between gravity and a coherent state of matter, such as a Bose-Einstein Condensate, leads to the fundamental question of whether it is possible that dark matter is in a coherent state, while the only interaction with ordinary matter is gravitational. Penrose has also used the Schrödinger-Poisson (SP) system of equations during his "Quantum State reduction" research programme (Brook and Coles, 2022). Presuming that DM does indeed consist of a BEC, it should be described by the coupled nonlinear Schrödinger-Poisson system.

The study of the light deflection by galaxies and gravitational lensing provides a powerful method for discriminating between the Bose–Einstein condensate dark matter model and other hypotheses concerning dark matter. Certain basic theoretical tools that may be used to obtain a more in-depth comparison between predictions made by the DM condensate model and observational data on the galactic rotation curves were provided by Boehmer and Harko (2007).

A scalar field described by the galactic dark matter, proposed by Mielke *et al.* (2003), corresponds to a gravitationally confined Bose-Einstein Condensate, but of galactic dimension. Thus, a cold Boson star can be also considered as a viable astrophysical object. In Chavanis and Harko (2012), the properties of the boson stars were investigated in both the non-relativistic and relativistic approaches, by considering a condensate with quartic non-linearity.

The essential analogy between dark matter's superfluidity and cold atom systems has been investigated by Berezhiani and Khoury (2015), emphasizing the importance of further cosmological investigations of the theory.

However, in many physical situations, such as higher densities, it is unlikely that the approximation of the dilute bosonic systems suffices; hence, one might need models which would account for long-range correlations, and multi-body interactions. Hence, in order to describe bosonic condensates in realistic physical situations, one must go beyond the quadratic self-interaction, and consider more general forms of the potential, and of the Gross-Pitaevskii equation. One of these possibilities that could open some new avenues for the understanding of the dark matter is the Gross-Pitaevskii equation with the logarithmic nonlinearity.

The first formulation of a fundamentally nonlinear equation that emphasized the necessary separability of non-interacting states is the logarithmic Schrödinger equation proposed by Bialynicki-Birula and Mycielski (1976). Multi-body interactions have been discovered to be a suitable attribute for the logarithmic Bose liquids, implying new degrees of freedom to rise (Zloshchastiev, 2012). Comparison between the behavior of two-body and three-body interactions demonstrates that the stability in the logarithmic BEC predominates the classical one (Bouharia, 2015).

There exist many physical situations where the logarithmic nonlinearity plays a major role in quantum systems. Astrophysical data observed from cosmic rays and the deformed vacuum wave dispersion have been explained by the non-trivial linearity (Zloshchastiev, 2012). Various further applications of the LogSE have been published in areas such as stochastic quantum mechanics (Lemos, 1980), nuclear physics (Kartavenko *et al.*, 1998), quantum optics (Buljan *et al.*, 2003), quantum liquids and superfluidity (Avdeenkov and Zloshchastiev, 2011; Bouharia, 2015; Zloshchastiev, 2012, 2018), and in the field of gravity (Zloshchastiev, 2010, 2011).

The energy additivity, as well as the separability of non-interacting subsystems in non-relativistic quantum mechanics are features not influenced by the logarithmic nonlinearity (Zloshchastiev, 2012). Nevertheless, the influence of the logarithmic nonlinearity on BEC may suggest regions with the equilibrium of the self-interaction and distribution of the halo which leads to solitonic behavior of the logarithmic BEC (Vowe *et al.*, 2020).

In the central potential and the non-zero angular momentum cases, the logarithmic SE maintains certain symmetries found in the linear counterpart (Shertzer and Scott, 2020). Logarithmic nonlinear systems should arise with a hydrodynamic description as an universal property. Extensive properties of the LogSE are the Galilean invariance, dimensional homogeneity, and the conservation of the norm. Quantum entanglement can not be produced by the nonlinearity of the system, and thus the logarithmic equation assures the separation of the time evolution from the product states.

Bose-Einstein Condensates with logarithmic nonlinearity have been also considered as a description of the properties of dark matter. A generalized Gross-Pitaevskii equation, including a logarithmic term, was obtained by using the theory of scale relativity, in Chavanis (2017) and Chavanis (2018), respectively.

The Jeans instability of the dark matter halos, which are described by a generalized Schrödinger equation, derived from the theory of scale relativity, and involving a logarithmic non-linearity, associated with an effective temperature and a source of dissipation, was considered in Ourabah (2020a). By using the Madelung transformation, the generalized Schrödinger equation can be reformulated in the form of a quantum hydrodynamic model which, after coupling to the Poisson equation, is used to study the Jeans gravitational instability. Two cases were considered, the first being a static and uniform unperturbed background. This case was generalized by including the effects of the expansion of the Universe on the zeroth-order dynamics. The stability of the dark matter halo is ensured by the effective temperature, and the nonlocality effects acting against gravity. On the other hand, the dissipation source damps the density contrast evolution, but does not modify the threshold value of the Jeans wave number.

The Benjamin-Feir type modulational instability of the nonlinear Schrödinger equation of scale relativity with a logarithmic nonlinearity was investigated in Ourabah (2020b). The equation was further generalized to include the presence of short-range interactions, thus leading to Gross-Pitaevskii and Cahn-Hilliard type equations, as well as to a generalization emerging from the Lynden-Bell distribution. A criterion for the modulational instability of the logarithmic condensate dark matter was established, and the corresponding growth rate was studied. For a recent discussion of the properties of the logarithmic Bose-Einstein Condensates, and for their relevance for dark matter, and modified gravity studies, see Ourabah (2023).

It is the main goal of the present manuscript to consider another important property of the logarithmic Bose-Einstein Condensates, namely, the physical characteristics of their gravitational collapse, and the properties of the objects formed when the collapse stops. We consider a logarithmic Schrödinger equation that includes both the terms of the ordinary SE, and a nonlinear term of the form $b\ln\frac{|\Psi(\vec{r},t)|^2}{|\Psi_0|^2}$, where b is the strength of the logarithmic nonlinearity in units of energy. A thermodynamical conjugate has been found to correlate the nonlinear coupling behavior b with a specific quantum temperature referred to by Zloshchastiev (2018). In this study the presence of cold quantum liquids and gases, such as BECs, has also been explained, as well as the empirical non-observability of logarithmic effects in some systems, and their predominance in others.

In order to obtain a clear physical picture of the collapse process, we introduce for the logarithmic Gross-Pitaevskii equation the Madelung representation of the wave function, which allows to represent the dynamical behavior of the condensate in terms of a continuity, and Euler type fluid mechanical equation. Moreover, we use the Thomas-Fermi approximation, which allows to neglect the effects of the quantum potential in the Euler equation. Furthermore, we assume that the collapse is spherically symmetric. By introducing a specific ansatz for the fluid velocity and density, and by considering a first order approximation of the logarithmic nonlinear term, the exact solutions of the Poisson equation and of the hydrodynamical equations are obtained, in a parametric form, with the solution depending on the nonlinear interaction strength b, the boson mass m, and total mass of the condensate. As an astrophysical application of the obtained results we estimate the formation time of astrophysical objects with masses in the range of $10^6 M_\odot$ and $10^{12} M_\odot$, which could represent either theoretical alternatives to the supermassive black holes at the galactic center, or can model the dark matter galactic halos.

The present paper is organized as follows. We introduce the basic theoretical model of the Bose-Einstein Condensate with logarithmic nonlinearity, as well as its hydrodynamical representation, in Section 2. The hydrodynamical evolution equations are written down in spherical symmetry in Section 3. By adopting two assumptions on the fluid velocity and density, the equation describing the time evolution of the radius of the condensate is obtained. The exact solution of the collapse equation is obtained in an integral form in Section 4. We discuss some astrophysical applications of the obtained results in Section 5. Finally, we discuss, and conclude our work in Section 6. The details of the derivation of the hydrodynamic representation of the Gross-Pitaevskii equation are given in Appendix A.

2. BOSE-EINSTEIN CONDENSATE WITH LOGARITHMIC NONLINEARITY

The general nonlinear Schrödinger equation (also known as Gross-Pitaevskii equation) used to describe the time evolution of the dilute BEC is given by (Dalfovo *et al.*, 1999; Strigari, 2013; Boehmer and Harko, 2007)

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+mV_{ext}+g'\left(|\Psi(\vec{r},t)|^2\right)\right]\Psi(\vec{r},t), \tag{8}$$

where m is the mass of the condensate particle, V_{ext} is the external potential, which we will assume to be the gravitational potential, while the term $g'(|\Psi|^2)$, introduced in Barcelo *et al.* (2001), describes the self-interaction of the particles in the system, with a prime denoting the derivative with respect to the independent variable.

As for the logarithmic BEC, the self-interaction term has the form (Bialynicki-Birula and Mycielski, 1976; Zloshchastiev, 2022),

$$g'\left(|\Psi(\vec{r},t)|^2\right)=b\ln\frac{|\Psi(\vec{r},t)|^2}{|\Psi_0|^2}, \tag{9}$$

where b and Ψ_0 are constants. The normalization condition requires that $N=\int|\Psi(\vec{r},t)|^2 d^3\vec{r}$, where N represents the total number of the particles in the condensate.

2.1. HYDRODYNAMIC REPRESENTATION AND THE EQUATION OF STATE OF THE LOGARITHMIC BEC

Hence, the Gross-Pitaevskii Eq. (8), in the presence of a logarithmic self-interaction term, takes the form (Bialynicki-Birula and Mycielski, 1976; Zloshchastiev, 2012, 2022),

$$i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}=\left[-\frac{\hbar^2}{2m}\nabla^2+mV_{ext}+b\ln\frac{|\Psi(\vec{r},t)|^2}{|\Psi_0|^2}\right]\Psi(\vec{r},t). \tag{10}$$

Based on Eq. (10), the quantum pressure of the logarithmic BEC dark matter can be worked out. Firstly, the wave function of the BEC dark matter with logarithmic nonlinearity is represented in polar form as

$$\Psi(\vec{r},t)=\sqrt{n(\vec{r},t)}e^{i\frac{S(\vec{r},t)}{\hbar}}, \tag{11}$$

where $n(\vec{r},t)=|\Psi(\vec{r},t)|^2$ is the particle number density in the system, and $S(\vec{r},t)$ is a phase factor (Dalfovo *et al.*, 1999). This form of the wave function leads to the Madelung representation of quantum mechanics (Madelung, 1926). We also introduce the mass density of the condensate, defined as $\rho_m(\vec{r},t)=mn(\vec{r},t)=m|\Psi(\vec{r},t)|^2$. With the wave function $\Psi(\vec{r},t)$ replaced by Eq. (11), Eq. (10) becomes

$$i\hbar\frac{\partial}{\partial t}\left[\sqrt{n(\vec{r},t)}e^{i\frac{S(\vec{r},t)}{\hbar}}\right]=\left[-\frac{\hbar^2}{2m}\nabla^2+U(\vec{r},t)\right]\sqrt{n(\vec{r},t)}e^{i\frac{S(\vec{r},t)}{\hbar}}, \tag{12}$$

where

$$U(\vec{r},t)=mV_{ext}(\vec{r},t)+b\ln\frac{|\Psi(\vec{r},t)|^2}{|\Psi_0|^2}. \tag{13}$$

After we separate the real and imaginary parts of Eq. (12), and simplify both of them, we obtain (for the details of the calculation see 6)

$$\frac{\partial\rho_m}{\partial t}+\nabla\cdot(\rho_m\vec{v})=0, \tag{14}$$

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{m}\nabla(Q+U), \tag{15}$$

where $\vec{v}=\nabla S/m$ is the velocity of the quantum fluid and

$$Q=-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho_m}}{\sqrt{\rho_m}}, \tag{16}$$

is the Bohm quantum potential (Dalfovo *et al.*, 1999; Strigari, 2013).

From a physical point of view, Eq. (14) can be interpreted as the continuity equation of the quantum fluid. As for Eq. (15), it has the same form as the Euler equation in standard Newtonian fluid mechanics. In the following we will adopt the Thomas-Fermi approximation, according to which the quantum potential Q can be neglected, if the number of particles in the system is large enough (Dalfovo *et al.*, 1999; Harko, 2011a). Hence Eq. (15) takes the form

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{m}\nabla U=-\nabla V_{ext}-\frac{b}{m}\nabla\ln\frac{\rho_m}{\rho_0}, \tag{17}$$

where $\rho_0=m|\Psi_0|^2$. Equivalently, the above equation can be written as

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{m}\nabla U=-\nabla V_{ext}-\frac{b}{m}\frac{1}{\rho_m}\nabla\rho_m. \tag{18}$$

Considering the logarithmic BEC as an inviscid fluid, we can compare Eq. (18) with the Euler equation in fluid dynamics,

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=\vec{F}-\frac{\nabla P}{\rho_m}, \tag{19}$$

where $\vec{F}=-\nabla V_{ext}$ is the external force term, ρ_m is the mass density, and p is the thermodynamic pressure. Hence, by comparing the two Euler type equations, it is easy to find the mathematical form of the quantum pressure of the logarithmic BEC as given by the linear function (Chavanis, 2018; Ourabah, 2023),

$$P=\frac{b}{m}\rho_m. \tag{20}$$

Therefore, the BEC with logarithmic nonlinearity satisfies the ideal gas equation of state, with its thermodynamic pressure being proportional to the fluid density.

However, since the hydrodynamical evolution equations for a classical fluid are rather complicated, and generally they can be solved only numerically, in order to simplify the mathematical formalism we will introduce at this moment an approximate formulation of the Euler equation. We adopt as an exact equation of state of the dark matter BEC the ideal gas equation of state, Eq. (20). At the level of the exact Eq. (17), we approximate the logarithmic term as $\ln\frac{\rho_m}{\rho_0}=\frac{\rho_m}{\rho_0}-1$. Then Eq. (17) becomes

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{m}\nabla U=-\nabla V_{ext}-\frac{b}{m}\frac{\nabla\rho_m}{\rho_0}, \tag{21}$$

and

$$\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}=-\frac{1}{m}\nabla U=-\nabla V_{ext}-\frac{1}{\rho_0}\nabla P. \tag{22}$$

Within this approximation, the hydrodynamical evolution is also determined by an arbitrary constant density term ρ_0, which can be interpreted as an effective average density of the condensate.

3. HYDRODYNAMICAL EVOLUTION EQUATIONS

We consider now the gravitational collapse of the logarithmic Bose-Einstein dark matter condensate. For the spherically symmetric collapse with $\vec{v} = (v_r, 0, 0)$, by using the Thomas-Fermi approximation, the Poisson equation $\nabla^2\Phi = 4\pi G\rho_m$, Eqs. (14) and (15) take the form

$$\frac{1}{\rho_m}\frac{\mathrm{d}\rho_m}{\mathrm{d}t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = 0, \tag{23}$$

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{1}{\rho_0}\frac{\partial P}{\partial r} + \frac{\partial \Phi}{\partial r} = 0, \tag{24}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 4\pi G\rho_m, \tag{25}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r}$ and $P = \frac{b}{m}\rho_m$.

3.1. EXACT SOLUTION OF THE HYDRODYNAMICAL EQUATION

In order to obtain an exact solution of the hydrodynamic equations (23)-(25) we will adopt the following ansatz for the velocity and density of the condensate,

$$v_r(r,t) = \frac{\dot{R}(t)}{R(t)}r, \rho_m(r,t) = \frac{\alpha}{R^3(t)} + \frac{\beta r^2}{R^5(t)}, \tag{26}$$

where α and β are constants, and $R(t)$ is the radius of the condensate at time t. By substituting the above expression into the continuity Eq. (23) it follows that it is identically satisfied for all values of α and β. The Poisson equation (25) can be integrated to give first

$$\frac{\partial\Phi(r,t)}{\partial r} = \frac{1}{r^2}\left[\Phi_0(t) + 4\pi G\left(\frac{\alpha r^3}{3R^3(t)} + \frac{\beta r^5}{5R^5(t)}\right)\right], \tag{27}$$

where $\Phi_0(t)$ is an arbitrary function of integration. In order to avoid a singularity at the center we take $\Phi_0 = 0$. A further integration gives for the gravitational potential the expression

$$\Phi(r,t) = \Phi_1(t) + 4\pi G\left[\frac{\alpha r^2}{6R^3(t)} + \frac{\beta r^4}{20R^5(t)}\right], \tag{28}$$

where $\Phi_1(t)$ is an arbitrary function of integration. The vacuum boundary of the condensate is defined by the condition $\rho(r,t)|_{r=R(t)} = 0$, which requires $\beta = -\alpha$, giving for the density of the condensate the expression

$$\rho_m(r,t) = \frac{\alpha}{R^3(t)}\left(1 - \frac{r^2}{R^2(t)}\right). \tag{29}$$

The total mass of the dark matter halo is given by

$$M = \int_0^{R(t)} \rho_m(r,t) 4\pi r^2 \, dr = \frac{8\pi\alpha}{15}, \tag{30}$$

giving $\alpha = \frac{15M}{8\pi}$. At the surface of the condensate $\Phi(r,t)|_{r=R(t)} = -\frac{GM}{R(t)}$, giving

$$\Phi_1(t) = -\frac{15MG}{8R(t)}. \tag{31}$$

Hence we obtain the gravitational potential as

$$\Phi(t) = -\frac{15MG}{8R(t)}\left[1 - \frac{2r^2}{3R^2(t)} + \frac{r^4}{5R^4(t)}\right]. \tag{32}$$

The Euler equation (24) becomes

$$\frac{\ddot{R}(t)}{R(t)} r - \frac{2\alpha b}{\rho_0 m R^5(t)} r + \frac{4\pi G\alpha}{R^3(t)} r \left[\frac{1}{3} - \frac{r^2}{5R^2(t)}\right] = 0. \tag{33}$$

In the last term of the above equation we approximate r^2 by $R^2(t)$, and thus we obtain for $R(t)$ the following equation of motion,

$$\ddot{R}(t) - \frac{2b\alpha}{\rho_0 m R^4(t)} + \frac{8\pi G\alpha}{15R^2(t)} = 0. \tag{34}$$

By multiplying the above equation by $\dot{R}(t)$ we obtain the first integral,

$$\frac{1}{2}\dot{R}^2(t) + \frac{2b\alpha}{3R^3(t)\rho_0 m} - \frac{8\pi G\alpha}{15R(t)} = E_0, \tag{35}$$

where E_0 is an arbitrary constant of integration. Hence, the problem of the collapse of the logarithmic Bose-Einstein Condensate has been reduced to the problem of the motion of a point particle in the effective potential

$$V_{eff}(R) = \frac{2b\alpha}{3R^3\rho_0 m} - \frac{8\pi G\alpha}{15R}. \tag{36}$$

The equilibrium positions of the system can be obtained from the condition $\partial V_{eff}(R)/\partial R = 0$.

4. COLLAPSE OF THE CONDENSATE

In order to attain the collapsed time dependent on the radius of the dark matter halo, we proceed to use the dimensionless method. Consequently, we replace the time and radius variables by $t = \gamma\tau$ and $R(t) = \delta\theta(t)$, where τ and θ are the dimensionless parameters. After imposing the conditions $\frac{2b\alpha\gamma^2}{3\delta^5\rho_0 m} = 1$ and $\frac{8\pi G\alpha\gamma^2}{15\delta^3} = 1$, Eq. (35)

becomes

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2+\frac{1}{\theta^3}-\frac{1}{\theta}=\varepsilon, \tag{37}$$

where $\varepsilon=\frac{\gamma^2}{\delta^2}E_0$.

With $\rho_0=3M/4\pi\delta^3\theta_{eq}^3$, the constants δ and γ have the expression

$$\delta=\frac{G}{5\sqrt{3}}\frac{Mm}{b}, \quad \gamma=\left(\frac{1}{5\sqrt{3}}\right)^{3/2}MG\left(\frac{m}{b}\right)^{3/2}. \tag{38}$$

The dimensionless effective potential is

$$V_{eff}(\theta)=\frac{1}{\theta^3}-\frac{1}{\theta}. \tag{39}$$

A plot of the potential is shown in Figure 1 where the equilibrium value of $\theta_{eq}=\sqrt{3}$ is apparent.

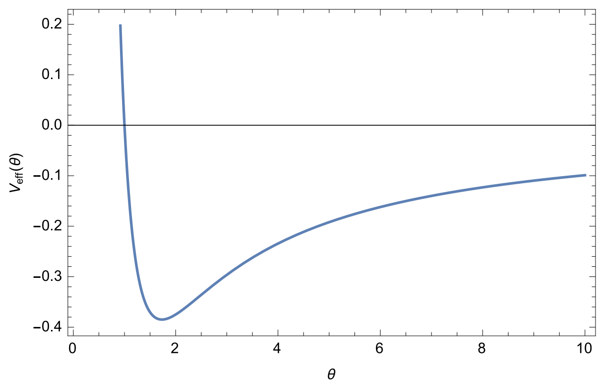

Fig. 1 – The dependence of the dimensionless effective potential on dimensionless radius.

Formally, the dimensionless collapse time can be obtained from (37) as

$$\tau=-\int_{\theta_0}^{\theta_{eq}}\frac{1}{\sqrt{2\left(\varepsilon-\frac{1}{\theta^3}+\frac{1}{\theta}\right)}}d\theta, \tag{40}$$

where θ_0 is the initial radius, and the ”–” sign corresponds to the contraction of the halo, (the ”+” sign corresponds to the condensate expansion),

As the kinetic energy is always positive, the effective potential expression clearly indicates that ε may only take values greater than a certain minimum $\varepsilon_{min}=V_{eff}(\sqrt{3})$. From (37), $\varepsilon=\frac{1}{\theta_0^3}-\frac{1}{\theta_0}$. In terms of the initial radius, the collapse time becomes

$$\tau = -\int_{\theta_0}^{\theta_{eq}} \frac{1}{\sqrt{2\left(\frac{1}{\theta_0^3} - \frac{1}{\theta_0} - \frac{1}{\theta^3} + \frac{1}{\theta}\right)}} d\theta. \tag{41}$$

A plot of $\tau(\theta_0)$ is shown in Figure 2.

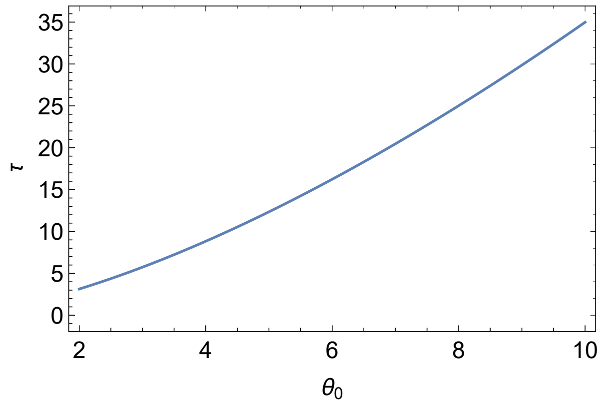

Fig. 2 – The dependence of the dimensionless collapse time on the dimensionless initial radius of the DM halo.

5. ASTROPHYSICAL APPLICATIONS

After the value of the dimensionless collapse time as function of dimensionless initial radius is obtained with (41) it is easy to return to the space $(t, R(t))$ via (38). The collapse time is parameterized by the interaction strength b, the mass of the particle m and the overall condensate mass M, where for the numerical values of the particle's mass we adopt the values proposed in Delgado and Mateo (2023). From the analysis of (38), it is evident that the parametrization is nonlinear, and as such each of these parameters will leave a nontrivial imprint in observational data.

The effect of each of the parameters in the set $\{b, m, M\}$ on the relationship between collapse time and initial radius of the condensate is investigated in Figures 3–5, where for the order of magnitude of the initial radius values we adopt the estimations introduced in Gavrilik and Nazarenko (2021). Each point on these plots represents a result that can be easily compared to observations. As an example, Table 1 contains numerical results for a boson mass $m = 5.6 \cdot 10^{-23}$ eV, a BEC with central density $\rho_c = 3 \cdot 10^{-22}$ kg/m^3, and pressure $P_0 = 10^{-12}$ Pa. The general trend is that collapse time increases with larger initial radii of the condensate.

Within this general trend, the collapse time increases with increasing boson self-interaction. A spread of 14 orders of magnitude in self interaction strength is

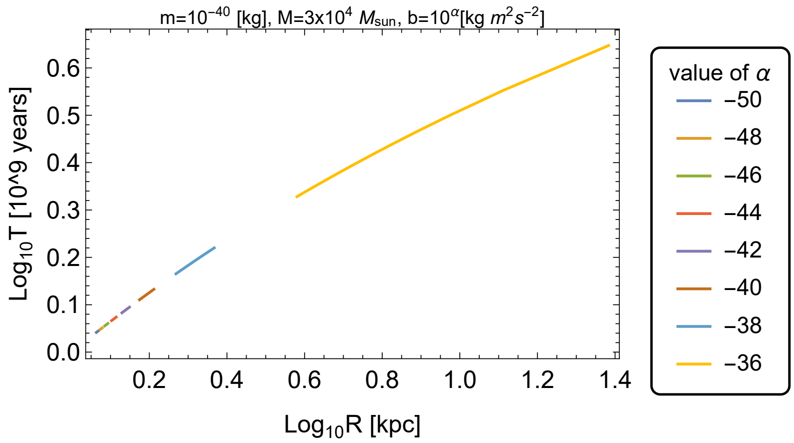

Fig. 3 – Collapse time-radius relation for a fixed condensate mass M and boson mass m. The colors stand for different exponent α in the scaling of the interaction strength $b = 10^{\alpha}$.

Table 1

The initial radius and collapse time determined for different masses of DM halos

Mass	Initial radius (kpc)	Collapse Time (10^9 years)
$2.2 \cdot 10^6 M_{\odot}$	0.02	0.0012
$2.5 \cdot 10^7 M_{\odot}$	0.3	0.014
$4.8 \cdot 10^8 M_{\odot}$	0.15	0.001
$2 \cdot 10^9 M_{\odot}$	0.6	0.01
$3.4 \cdot 10^9 M_{\odot}$	2.7	0.038
$1.2 \cdot 10^{10} M_{\odot}$	38	1.1
$10^{12} M_{\odot}$	317	2.8

mapped into at most one order of magnitude in the collapse time, as shown in Figure 3.

For fixed interaction strength and initial condensate mass, a spread of 24 orders of magnitude in boson mass leads to a change of two orders of magnitude in the collapse time, as indicated by the results of Figure 4.

For fixed interaction strength and boson mass, a spread of 10 orders of magnitude in condensate mass leads to a change within order of magnitude in collapse time, as shown in Figure 5.

These results show that a large variety of stable cosmic structures can form as a result of a logarithmic BEC, ranging from supermassive black holes at the galactic centers, to galactic halos. The collapse time is strongly dependent on the mass of the dark matter particle, and of the interaction strength, which must be determined either from fundamental physics considerations, from experimental investigations performed in laboratory, or by using some observational data on astrophysical or

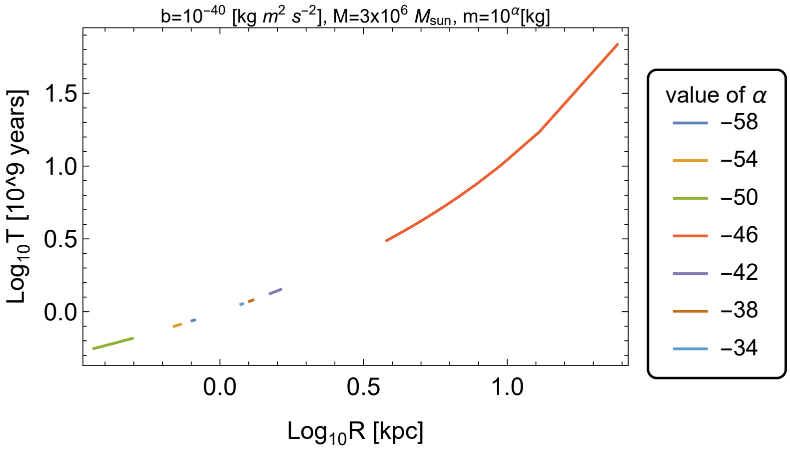

Fig. 4 – Collapse time-radius relation for a fixed condensate mass M and interaction strength b. The colors stand for different exponent α in the scaling of the boson mass $m = 10^{\alpha}$.

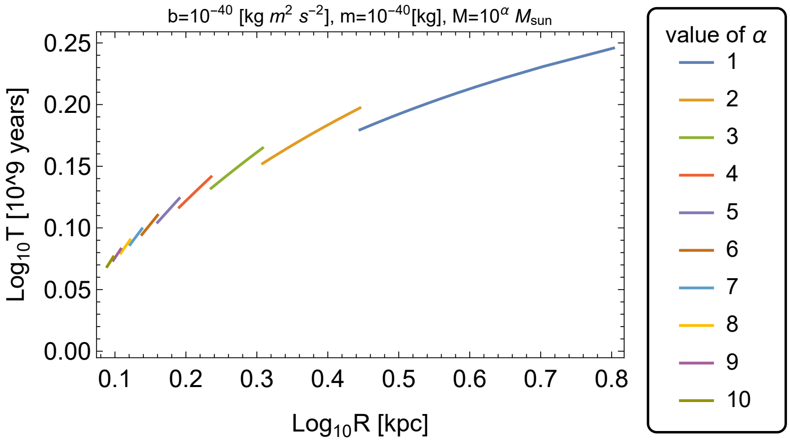

Fig. 5 – Collapse time-radius relation for a fixed interaction strength b and boson mass m. The colors stand for different exponent α in the scaling of the condensate mass $M = 10^{\alpha} M_{\odot}$.

cosmological scales. It is also important to note that the collapse time for the formation of a super-massive object of mass of around $10^9 M_\odot$ is relatively short in the present model, being of the order of 10^7 years, which is smaller than the age of the quasar J0313–1806, which is of the order of 6×10^8 years (Wang *et al.*, 2021a). Hence, the possibility of the early formation of supermassive objects in the early Universe due to the rapid collapse of logarithmic Bose-Einstein Condensates may provide an attractive explanation for the existence of very luminous quasars, formed almost immediately after the Big Bang. Of course, to obtain a full picture of the collapse of the logarithmic BECs, a complete general relativistic analysis is needed, and this analysis may modify some essential characteristics of the present Newtonian analysis.

6. CONCLUSIONS

In the present paper we have investigated some aspects of the gravitational collapse of a logarithmic Bose-Einstein condensate, described by the coupled system of the nonlinear Gross-Pitaevskii and Poisson equations. By using the hydrodynamic representation of the GPP system, by adopting a particular form of the fluid velocity and density, and after some first order approximations performed in the equations of motion, a parametric exact solution of the continuity and Euler equations can be obtained. The solutions allows for the possibility of estimating the formation time of astronomical objects with masses ranging from $10^6 M_\odot$ to $10^{12} M_\odot$.

The gravitational collapse of Bose-Einstein Condensates with quartic nonlinearity was considered in Harko (2014) and Harko (2019). In Harko (2014), the collapse was studied by using a variational approach. With the help of an appropriately chosen trial wave function, the Gross-Pitaevskii equation can be reformulated in spherical symmetry as Newton's equation of motion for a particle in an effective potential. The collapse of the Bose-Einstein Condensate dark matter reaches a final point once a stable configuration is formed, corresponding to the minimum of the effective potential. The variational approach allows for obtaining the radius and the mass of the resulting dark matter object, together with the collapse time of the condensate. The critical scales above which condensate dark matter collapses are given by the Jeans radius and the Jeans mass, respectively, and they have been discussed in detail in Harko (2019). The collapse/expansion of the rotating condensed dark matter halos was also investigated, and a family of exact semi-analytical solutions of the hydrodynamic evolution equations was found. The solution was obtained by using the method of separation of variables. An approximate first order solution of the hydrodynamic equations describing the condensate was also investigated.

The stability and free expansion of a one-dimensional logarithmic Bose–Einstein condensate was investigated in Rodríguez-López and Castellanos (2021), by using a variational approach. The free velocity expansion also shows significant differences with respect to the three dimensional system, once the logarithmic interactions are considered. The one dimensional logarithmic condensate tends to form quantum droplet-type configurations. Moreover, the cloud oscillates during the free expansion around a specific equilibrium size, thus leading to the formation of oscillating quantum droplets.

Condensed object could exist in large variety of forms, ranging from small mass stars to huge galactic halos. Stellar type objects having a dominant bosonic component (boson stars) are also of considerable astrophysical importance.

Boson stars are hypothetical objects, composed of boson that obey the Bose-Einstein statistics, and are kept together by the gravitational force, balanced by the quantum pressure of the component particles (Cunha *et al.*, 2022; Herdeiro *et al.*, 2022; Sanchis-Gual *et al.*, 2022). However, when the mass of the boson star surpasses the critical value referred to as the Chandrasekar limit, it collapses under its own gravity. The time it takes for a boson star to undergo gravitational collapse is dependent on various parameters, including its initial mass, size, and composition. Attaining a stable radius after the collapse of a boson star is a conceivable scenario, albeit several factors such as the characteristics of the constituent bosons (mass and spin). Boson stars are considered as one of the possible alternatives for massive stellar type objects, whose properties cannot be explained by the standard neutron star models (Sin, 1994; Brook and Coles, 2022; Boehmer and Harko, 2007; Mielke *et al.*, 2003; Harko, 2011a; Chavanis and Harko, 2012), and may be present in the early universe as a result of the condensation of axions. The density profile of atoms in a logarithmic or nonlinear Bose-Einstein condensate deviates from a power-law scaling, and instead demonstrates a logarithmic scaling. As a result, for a logarithmic nonlinearity, a different form of behaviour emerges as compared to that of a typical BEC with quadratic self-interaction.

The present paper has explored the effect of a nonlinear term, of logarithmic form, on the collapse of a Bose-Einstein Condensate dark matter halo. To describe the condensate, the Gross-Pitaevskii and the Poisson equations were adopted, with the logarithmic nonlinearity included in the GP equation. With the use of the hydrodynamic representation of the Gross-Pitaevskii equation, the equation of state of the dark matter condensate was obtained. Interestingly enough, this equation of state takes the form of the ideal gas equation of state, with the pressure proportional to the density of the condensate.

In order to obtain an analytical description of the collapse, we have introduced several approximations, and simplifying assumptions. First, after adopting the equation of state of the condensate, we have introduced a first order approximation of the logarithmic nonlinear term. Secondly, we have assumed some specific forms for the velocity field and density distribution of the condensate, by assuming for the radial velocity a strict proportionality with the radial coordinate, with the proportionality factor a function of time only. With the help of this specific form of the velocity, the equation of continuity of the fluid fixes the analytic expression of the density, as the sum of two terms, the first being a function of time only, independent on r, while the second term is proportional to r^2. With the help of these assumptions, the Poisson equation for the gravitational potential can be integrated exactly, while the Euler equation reduces to a second ordinary differential equation for the radius $R(t)$ of the condensate. The solution of the Euler equation can be obtained in the form of quadratures, after a set of dimensionless physical quantities has been introduced. To the best knowledge of the authors, the presented solution has not been previously obtained in the literature on the logarithmic Bose-Einstein Condensates.

From an astrophysical point of view we have investigated the possibility of the formation of various objects of different masses. For several classes of massive objects, with masses in the range $\left(10^6 M_\odot, 10^{12} M_\odot\right)$, we have estimated the collapse time, as well as the initial radius of the condensate, before the onset of instability.

The present results indicate that the presence of gravitational instabilities in logarithmic Bose-Einstein Condensate dark matter can lead to the formation of a large variety of astrophysical objects, with masses ranging in the planetary and supermassive black hole range. The parameters dictating the behavior of the collapse time are the nonlinear interaction strength, the dark matter particle mass (assumed to be a boson), and the total condensate mass. The set of all these values was numerically mapped, and are readily comparable with observations. Hence, logarithmic Bose-Einstein Condensates may prove to be a viable candidate for dark matter. On the other hand, they may also represent some attractive explanations for the existence of the superrmassive black holes at the galactic center, which may have been formed as the result of the collapse of a dark cloud Bose-Einstein Condensate.

We would like to point out that our analysis is mostly qualitative, and that a full understanding of the collapse of a logarithmic Bose-Einstein Condensate requires the extensive use of numerical methods to integrate the hydrodynamic evolution equations. Moreover, for a realistic description of the very massive objects at the galactic centers, a full general relativistic approach must be also implemented. Our present results are a first step in these directions. Even so, they still point towards the rich and complex dynamical behavior of the Bose-Einstein Condensates with logarithmic nonlinearity, and they also provide some basic theoretical tools for the understanding of galactic structure, and evolution.

APPENDIX. THE HYDRODYNAMICAL FORMULATION OF THE GROSS-PITAEVSKI EQUATION

In this Appendix, we present the details of the calculations leading to the hydrodynamic formulation of the Gross-Pitaevskii equation with a logarithmic self-interaction. Starting from the equation (3), after substituting the representation of the wave function as given by Eq. (4), we first obtain

$$i\hbar\frac{\partial}{\partial t}\left[\sqrt{n(\vec{r},t)}e^{i\frac{S(\vec{r},t)}{\hbar}}\right]=\left[-\frac{\hbar^2}{2m}\nabla^2+U(\vec{r},t)\right]\sqrt{n(\vec{r},t)}e^{i\frac{S(\vec{r},t)}{\hbar}}. \tag{42}$$

Then, after performing the differential operations, we find

$$\begin{aligned}i\hbar\left[\frac{\partial n}{\partial t}\frac{1}{2\sqrt{n}}e^{i\frac{S}{\hbar}}+\sqrt{n}e^{i\frac{S}{\hbar}}\frac{i}{\hbar}\frac{\partial S}{\partial t}\right]&=-\frac{\hbar^2}{2m}\nabla^2\left(\sqrt{n}e^{\frac{iS}{\hbar}}\right)+U\sqrt{n}e^{\frac{iS}{\hbar}}\\&=-\frac{\hbar^2}{2m}\nabla\cdot\left(\nabla\sqrt{n}e^{i\frac{S}{\hbar}}+\sqrt{n}e^{i\frac{S}{\hbar}}\frac{i}{\hbar}\nabla S\right)+U\sqrt{n}e^{i\frac{S}{\hbar}}\\&=-\frac{\hbar^2}{2m}(\nabla^2\sqrt{n}e^{i\frac{S}{\hbar}}+2\cdot\nabla\sqrt{n}e^{i\frac{S}{\hbar}}\frac{i}{\hbar}\nabla S-\sqrt{n}e^{i\frac{S}{\hbar}}\frac{|\nabla S|^2}{\hbar^2}+\\&+\sqrt{n}e^{i\frac{S}{\hbar}}\frac{i}{\hbar}\nabla^2S)+U\sqrt{n}e^{i\frac{S}{\hbar}}.\end{aligned} \tag{43}$$

Dropping the factor $e^{i\frac{S}{\hbar}}$ on both sides of Eq. (43), and separating the imaginary and real part of it, two different equations are obtained. The imaginary part of Eq. 43 is given by

$$\frac{\partial n}{\partial t}\frac{\hbar}{2\sqrt{n}}=-\frac{\hbar}{2m}\left(2\cdot\nabla\sqrt{n}\cdot\nabla S+\sqrt{n}\nabla^2S\right). \tag{44}$$

By letting $\vec{v}=\nabla S/m$, the above equation becomes,

$$\frac{\partial n}{\partial t}\frac{1}{2\sqrt{n}}=-\nabla\sqrt{n}\cdot\vec{v}-\frac{\sqrt{n}}{2}\nabla\cdot\vec{v}, \tag{45}$$

or, equivalently,

$$\frac{\partial n}{\partial t}=-\nabla n\cdot\vec{v}-n\nabla\cdot\vec{v}=\nabla\cdot(n\vec{v}), \tag{46}$$

finally giving

$$\frac{\partial\rho_m}{\partial t}=-\nabla\cdot(\rho_m\vec{v}), \tag{47}$$

which is the continuity equation in fluid dynamics.

As for the equation corresponding to the real part of Eq. (43), it can be successively transformed as

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\left(-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} + \frac{|\nabla S|^2}{2m} + U\right)\\ &= -\left(-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho_m}}{\sqrt{\rho_m}} + \frac{mv^2}{2} + U\right)\\ &= -(Q+K+U),\end{aligned} \tag{48}$$

where $Q = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho_m}}{\sqrt{\rho_m}}$ is the Bohm quantum potential, $K = \frac{mv^2}{2}$ is the kinetic energy, and U the overall potential energy. Replacing $\vec{v} = \nabla S/m$, Eq. (48) becomes

$$\begin{aligned}\frac{\partial \vec{v}}{\partial t} &= -\frac{1}{m}\nabla(Q+K+U)\\ &= -\frac{1}{m}\nabla(Q+U) - (\vec{v}\cdot\nabla)\vec{v},\end{aligned} \tag{49}$$

which is the quantum equivalent of the Euler equation in fluid mechanics.

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