A RIGOROUS ASTROMETRICAL SOLUTION IN THE CASE OF THE EULER – POINSOT EQUATIONS SYSTEM

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Abstract. The difference between the theory and the astronomical determinations regarding the latitude variations could be too great to be a real one! This subject we mentioned it in a previous paper (Ciobanu, 2008). Consequently, the current paper is meant to continue the study of this subject and to present a possible solution.

Key words: Celestial mechanics – Earth's rotation – Euler's theory .

One of the celestial mechanic's unresolved problems is the "O-C" (observational data minus theoretical calculus) difference regarding the rotation of the Earth around its axis. It is well known that the astronomical results regarding the latitude variations differ from values stipulated in the Euler-Poinsot theoretical case (Smart, 1953).

Obviously, the variation of the terrestrial coordinates – especially latitudes – is mainly due to the change of the rotational axis' position within the Earth. The first well known periodical result related to the observed latitude, is the Chandler period of approx. 14 months, which differs by four months from the Euler period (10 months), theoretically determined in the Euler-Poinsot case.

The dynamical equations regarding the Earth's rotation were established by Leonard Euler (1707 – 1783). He considered Earth as a rigid body with a fixed point. The fundamental equation of classical mechanics is in this case:

$$\mathrm{D}\overrightarrow{K}/\mathrm{D}t = \overrightarrow{M}_{fex}, \tag{1}$$

where $\overrightarrow{\mathrm{K}}$ is the kinetic momentum of the body with fixed point, $\mathrm{D}\overrightarrow{K}/\mathrm{D}t$ is the differential relative to the fixed frame of coordinates, $\overrightarrow{M}_{fex}$ is the momentum of the resultant of external forces, relative to fixed point.

In a relative movement, when the mobile axes Oxyz are fixed in the body and the inertial frame OXYZ has the origin also in "O", the relation (1) becomes:

$$\partial\overrightarrow{K}/\partial t + \overrightarrow{\varpi} \times \overrightarrow{\mathrm{K}} = \overrightarrow{M}_{fex}, \tag{2}$$

where $\overrightarrow{\varpi}$ represents the instantaneous rotation velocity which passes through the fixed point "O", and $\partial\overline{K}/\partial t$ is the differential of $\overrightarrow{K}$ relative to the mobile frame. Thus, we have:

$$\begin{aligned} A\dot{p} + (C-B)qr &= Mx \\ B\dot{q} + (A-C)rp &= My \\ C\dot{r} + (B-A)pq &= Mz \end{aligned} \tag{3}$$

Where A, B, C are the main momentums of inertia (the mobile axes thus coinciding with the main inertial axes), p, q, r are the $\overrightarrow{\omega}$ components in the mobile frame, and *Mx, My, Mz* are the $\overrightarrow{M}_{fex}$ components on the inertial frame.

Euler also gave the vectorial relations between $\overrightarrow{\omega}$ and its components in the terrestrial poles axis, inertial poles axis, and nodal axis (Fig. 1).

$$\overrightarrow{\omega} = \dot{\theta} versON + \dot{\varphi} versOz + \dot{\psi} versOZ, \tag{4}$$

where $\dot{\varphi}$ is the $\overrightarrow{\omega}$ component on the terrestrial poles axis, $\dot{\psi}$ is the $\overrightarrow{\omega}$ component on the inertial poles axis, and $\dot{\theta}$ is the component on the nodal line ON (the nutation velocity). Obviously φ, ψ and θ represent, respectively, the angles described by the own motion, precession and nutation, around the axes mentioned above (Fig. 1).

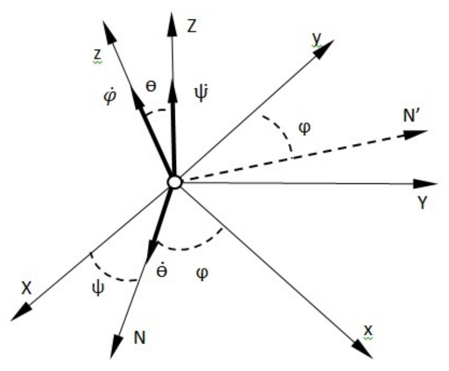

Fig. 1 – The position of the Euler's angles relative to the mobile axis (Oxyz) and to the inertial axis (OXYZ).

Consequently, the $\overrightarrow{\omega}$ components on the mobile frame (equatorial plane and geographical pole axes for a given epoch) are in Fig. 2.

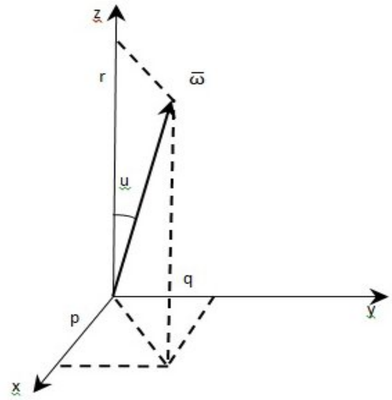

Fig. 2 – The components of the instantaneous rotational velocity (p,q,r) in the mobile frame (equatorial coordinates), where angle *u* marks a latitude variation.

$$\begin{aligned} p &= \dot{\theta}\cos(\varphi)+\dot{\psi}\sin(\theta)\sin(\varphi), \quad \text{on Ox axis} \\ q &= -\dot{\theta}\sin(\varphi)+\dot{\psi}\sin(\theta)\cos(\varphi), \quad \text{on Oy axis} \\ r &= \dot{\varphi}+\dot{\psi}\cos(\theta), \quad \text{on Oz axis.} \end{aligned} \tag{5}$$

The projection of $|\overrightarrow{\omega}|$ in the equatorial plane is given by the relation:

$$\sqrt{p^2+q^2}=\sqrt{(\dot{\theta})^2+(\dot{\psi})^2\sin^2(\theta)} \tag{6}$$

Formula (6) indicated the strong relation between a latitude variation and the precession and nutation phenomenon in the case of a model of rigid Earth.

It is known that the astronomical determinations for $\dot{\theta}$ and $\dot{\psi}$ gives very small values comparative with $\dot{\varphi}$, where φ indicated the diurnal sidereal time. Indeed in a sidereal time unit, $\varphi = 1296000''$, while $\psi < 0.2''$ and $\theta < 0.001''$.

As a result, the values given by (5) regarding coordinates p and q (which are related to the latitude variation) are extremely small. This is why only during the 20 century the variations of latitude where detected and systematical studied.

Therefore, if we note $\sqrt{p^2+q^2}$ as "$VLat$" (latitude variation), the relation (6) in a first approximation conducts to $VLat = 0.3794\dot{\psi}$, for $\theta(j2ooo.0) = 23^o26'21''$.

From celestial mechanics it is known that the precession and nutation phenomenon is caused by a variation of Earth's instantaneous rotation axis $\overrightarrow{\omega}$ inside the inertial frame (if $\overrightarrow{M}_{fex}$ is not null). But simultaneously, formula (6) proves that the same above mentioned axis $\overrightarrow{\omega}$ changes its position also in the mobile frame of axes, which induces a latitude variation.

Therefore accordingly to Euler's theory regarding the Earth's rotation about his axis, there are two distinct cases.

FIRST: if the resultant of external forces do not pass through the Earth's center (that means Mfex is not zero) apart of precession and nutation phenomenon exists also a real variation of latitude.

SECOND: If all external astronomical forces pass through the Earth's center, ($\overrightarrow{M}_{fex}$=0), no latitude variation exists as nor precession and nutation phenomena.

But in the Euler-Poison case it results than when $\overrightarrow{M}_{fex}$ =0, a latitude variation of 300 days take place.

Indeed, in his work, Louis Poinsot (1777–1859) studied system when $\overrightarrow{M}_{fex}$= 0 and the Earth's center of mass is supposed to be in the fixed point, and also the main moments of inertia concerning axes Ox and Oy are equal, $A = B$ (Melchior, Smart (1953)). Therefore, equation (1) becomes:

$$\mathrm{D}\overrightarrow{K}/\mathrm{D}t = 0 \tag{7}$$

The components in the mobile frame are:

$$\begin{aligned} A\dot{p} + (C-A)qr &= 0 \\ A\dot{q} + (A-C)rp &= 0 \\ C\dot{r} &= 0. \end{aligned} \tag{8}$$

The well known solutions are:

$$r = \text{constant}; \quad p\dot{p} + q\dot{q} = 0, \tag{9}$$

$$p^2 + q^2 = const., \tag{10}$$

a constant value which cannot be less than 0, resulting

$$p^2 + q^2 + r^2 = \text{constant}, \tag{11}$$

further the module of $|\overrightarrow{\varpi}|$ is approximated by r (his component on Oz), *i.e.*

$$|\overrightarrow{\varpi}| = r. \tag{12}$$

In scientific papers, formula (10) is always presented as $p^2 + q^2 > 0$ that means that $\overrightarrow{\varpi}$ describes a cone around Oz – the Euler's cone, free nutation, in ten months (the Euler's period). Therefore the aperture of that cone is not a zero one, but equal to the angle u (see Fig. 2).

The angle u marks a latitude variation, so $\cos(r/|\overrightarrow{\varpi}|)$ must be strictly less 1.

Therefore if we take p^2+ q^2 strictly > 0 then a real latitude variation exist.

In the same time, the equality $\left|\overrightarrow{\omega}\right| = \mathrm{r}$ must be carefully analyzed. Indeed the angle which may indicates a latitude variation is extremely small, but as in precession case, during time it may increase (the equinoctial point traversed in two millennia an entire zodiacal constellation). Putting $\left|\overrightarrow{\omega}\right| = \mathrm{r}$ we cancel the possibilities of existing any variation of latitude from even beginning. It must be noticed that in the case of high precision in astrometry a latitude variation cannot be neglected, no matter small value it is.

But formula (12) gives $\cos(r/\left|\overrightarrow{\omega}\right|) = 1$, that denotes no latitude variation.

Finally, if we admit $p^2 + q^2$ is strictly greater than 0, we admit a latitude variation, but if we approximate $\left|\overrightarrow{\omega}\right|$ by r no latitude variation exists. Therefore, $p^2 + q^2 > 0$ means a real latitude variation, meanwhile equality $|\overrightarrow{\omega}| = r$ does not admit any latitude variation.

In consequence we propose , $p^2 + q^2 = 0$ *, as a most rigorous solution in the Euler–Poinsot case. We must conclude that the cone aperture in the Euler–Poinsot case is zero and no Euler period exist.*

This solution is the only one which satisfies entirely formula (7) because if $\overrightarrow{M}_{fex} = 0$ it results that the kinetic moment K is a vector constant in both reference systems (fix and mobile) and no precession and nutation phenomenon exists, as nor latitude variation.

From a vectorial and geometrical point of view, when high precision is need, to identify a vector with one of his projection in a cartesian system of coordinate result null values for the others two coordinates.

Thus even mathematically, instead of having $\left|\overrightarrow{\omega}\right|^2 = p^2 + q^2 + r^2$ we have $r^2 = p^2 + q^2 + r^2$ and by consequence of formula (12), $p^2 + q^2 = 0$.

During the last century, periods of 14 months were found in terrestrial tides and also in studies regarding the dynamic of atmosphere (Hameed and Currie, 1989). Like this, the Chandler period is caused by a special position of external astronomical forces.

Continuing to analyze the Euler-Poinsot motion, the question arises: is the Euler cone aperture a nonzero one?

From relation (12), it results that the Euler cone becomes a straight line, as single component of $\overrightarrow{\omega}$ on the Oz axis. That means from beginning, no latitude variations, no Euler period, no free nutation exist.

Often, the Chanddler period were associated with a non-rigid body. Using Lagrange variables and a very simple model of a non-rigid Earth, the calculus regarding Chandler's period suffers a correction of about a very little fraction of a day and not four months (Ciobanu, 1991).

1. NOTE

Usually, the Euler–Poinsot case is associated with the dynamics of a gyroscope.

Indeed, there are some similarities such especially the great values of kinetic momentum and the existence of the fixed point inside. However, the gyroscope case admits an initial impulse, while the system Euler–Poinsot denies the existence of any impulse. In the gyroscope case, after some time, the vector of the kinetic momentum returns in the initial place. Instead, in the astronomical phenomenon of the Earth's rotation case, the angles are growing.

2. CONCLUSIONS

We must conclude that the Euler-Poinsot system does not admit any variation of latitude, or free nutation. Therefore, the period of Euler (10 months) cannot exist and, from theoretical point of view, the solutions in the Euler-Poinsot case must be revised in the case of high precision astrometry.

Thus the precession and nutation represent a real and well known astronomical phenomena, which is accompanied by latitude variations even in a solid rigid body.

In the real Earth case (considering it as not an absolute rigid body), the changes in terrestrial latitude, as a consequence of external astronomical forces, acts on mechanical and geological forces (Ruchin (1958); Berger (1988)).

During the period of 1900 – 2000, the astronomical data *International Earth Rotation and Reference Systems Service* (IERS) detected not only multiple periodical components of latitude variation, but also a secular displacement of terrestrial rotation axis inside the Earth of about one second of arc. A very hazardous extrapolation in the past, from Hiparch to the present, may suggest (according to Euler's theory) a displacement of about 300–400 m of the North Pole.

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