GENERALIZED REGULARIZATION

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Abstract. The regularization of the perturbed two-body problem was studied by Levi-Civita (1906). We generalized the coordinate transformation given by Levi-Civita (LC) to obtain a set of regularizing transformation. The advantage of study of many regularizing transformations for a given "near approach" problem in celestial mechanics is that we can give detailed information about the motion around the critical, singularity points. We applied the LC transformation and the generalized LC transformation of third degree in case of LEO satellites. We compared the obtained numerical results, and we emphasized the importance of the regularization study close to the Earth.

Key words: Celestial Mechanics, Regularization.

In Memoriam Rodica Roman, Vasile Mioc and Eugeniu Grebenikov

1. INTRODUCTION

The regularization is essential in space dynamics, the most importance have in the satellite motion. A similarity and a difference between the motion of natural and artificial celestial bodies is that close approaches can be happened, but in the second case is unavoidable (Szebehely, 1967; Boccaletti *et al.*, 1996).

When the distance between the bodies approaches to zero (at near collision), than the forces acting between particles approach to infinity, and this event produces sharp bends of the orbit. At collision the equations of motion shows singularities (Stiefel *et al.*, 1971).

The continuation of the orbit after collision is not feasible since the solution encounters the singularity present in the problem. Moreover, during a numerical integration, to overcome this difficulty is to use a small step length and many step of integration surrounding the close approach. The numerical precision after the collision will be worse, because the round-off and truncation errors.

In the regularization theory to eliminate the singularity we introduce independent variable (time transformation) and dependent variable (coordinate transforma-

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tion), which produce the slowing down phenomenon (Csillik, 2003).

We are interested to transform singular equations of motion into regular ones. First, we treat collision orbits in the physical plane, and second, the problem is regularized using the generalized LC regularizations (LCn), so we can handle orbits in the parametric plane.

Any mission of an artificial space vehicle implies close approach at the start at the end of destination. Occasionally, collisions with the planetary bodies may occur. The LCn equations of motion revealed to be effective for investigating the long-term behavior of perturbed two-body problems, for example, those used for studying the dynamics of artificial satellites.

When a relative orbit is designed using a very simplified orbit model, then the formation station keeping the control laws will need to continuously compensate for these modelling errors and burn fuel. This fuel consumption, depending on the modelling errors, could drastically reduce the lifetime of the spacecraft formation. Regularization methods reduce the numerical errors. Consequently, the artificial satellite orbit's regularization problem problem is very interesting from both the academic and practical point of views.

We give explicitly the regularized equations of motions, which we applied to study the Low Earth Orbits (LEO), while majority of artificial satellites are placed in LEO.

2. MOTION IN PHYSICAL AND PARAMETRIC PLANE

For simplicity, we consider in the following that the third body moves into the orbital plane. Denoting S_1 and S_2 the components of the binary system (with masses m_1 and m_2), the equations of motion of the test particle in the coordinate system xS_1y (physical plane) are

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} = x - \frac{q}{1+q} - \frac{x}{(1+q)r_1^3} - \frac{q(x-1)}{(1+q)r_2^3} \tag{1}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} = y - \frac{y}{(1+q)r_1^3} - \frac{qy}{(1+q)r_2^3} \tag{2}$$

where

$$r_1 = \sqrt{x^2+y^2}\ , \quad r_2 = \sqrt{(x-1)^2+y^2}\ , \quad q = \frac{m_2}{m_1}. \tag{3}$$

These equations have singularities in terms $1/r1$ and $1/r2$. This situation corresponds to collision of the test particle with S_1 and S_2 (Roman, 2011). If the test particle approaches very closely to one of the primaries, such an event produces large gravitational force and sharp bends of orbit. The removing of these singularities can be done by regularization.

Observation
At Szebehely (1967) the origin is located in the mass center μ of the binary system, and in our article the mass center q is located in the center of the most massive star S_1. In fact, many authors use the barycentric coordinate system, there are important books and articles in which the authors use the coordinate system with origin in the center of the most massive star (Kopal, 1978; Eggleton, 1983; Seidov, 2004). We denote the mass ratio with $q=\frac{m_2}{m_1}$ as Kopal (1978). Other authors use $\mu=\frac{m_2}{m_1+m_2}$ as parameter (Szebehely, 1967), it is easy to verify that $\mu=\frac{q}{1+q}$.

We will briefly present the well-known Levi-Civita regularization methods (Roman *et al.*, 2012). For the regularization of the equations of motion in the $(q_1 S_1 q_2)$ coordinate system $(q_i, p_i, i=\overline{1,2}$ are the generalized coordinates and momenta, $q_1 = x, q_2 = y, p_1 = \frac{\mathrm{d}q_1}{\mathrm{d}t} - q_2, p_2 = \frac{\mathrm{d}q_2}{\mathrm{d}t} + q_1)$, we shall introduce new variables Q_1 and Q_2, connected with the coordinates q_1 and q_2 by the Levi-Civita equations (Levi-Civita, 1906):

$$q_1 = Q_1^2 - Q_2^2,\ q_2 = 2Q_1Q_2 \tag{4}$$

Using the above Levi-Civitas coordinate transformation, the equations of motion of the restricted three-body problem becomes

$$\begin{aligned}
\frac{\mathrm{d}Q_1}{\mathrm{d}t} &= \frac{P_1}{D} + \frac{Q_2}{2} \\
\frac{\mathrm{d}Q_2}{\mathrm{d}t} &= \frac{P_2}{D} - \frac{Q_1}{2} \\
\frac{\mathrm{d}P_1}{\mathrm{d}t} &= \frac{P_2}{2} - \frac{2qQ_1}{1+q} - \frac{2}{1+q}\frac{Q_1}{r_1^2} - \\
&- \frac{2q}{1+q}\frac{Q_1(r_1-1)}{r_2^3} + \frac{(P_1^2+P_2^2)Q_1}{4r_1^2} \\
\frac{\mathrm{d}P_2}{\mathrm{d}t} &= -\frac{P_1}{2} + \frac{2qQ_2}{1+q} - \frac{2}{1+q}\frac{Q_2}{r_1^2} - \\
&- \frac{2q}{1+q}\frac{Q_2(r_1+1)}{r_2^3} + \frac{(P_1^2+P_2^2)Q_2}{4r_1^2}
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
D &= 4(Q_1^2+Q_2^2), \\
r_1 &= \sqrt{Q_1^2+Q_2^2}, \\
r_2 &= \sqrt{(Q_1^2+Q_2^2-1)^2+4Q_1^2Q_2^2},
\end{aligned}$$

Table 1

Harmonic and conjugate polinomial functions

n	$f_n(Q_1,Q_2)$	$g_n(Q_1,Q_2)$
$n=0$	1	0
$n=1$	Q_1	Q_2
$n=2$	$Q_1^2-Q_2^2$	$2Q_1Q_2$
$n=3$	$Q_1^3-3Q_1Q_2^2$	$3Q_1^2Q_2-Q_2^3$
$n=4$	$Q_1^4-6Q_1^2Q_2^2+Q_2^4$	$4Q_1^3Q_2-4Q_1Q_2^3$
$n=5$	$Q_1^5-10Q_1^3Q_2^2+5Q_1Q_2^4$	$5Q_1^4Q_2-10Q_1^2Q_2^3+Q_2^5$
$n=6$	$Q_1^6-15Q_1^4Q_2^2+15Q_1^2Q_2^4-Q_2^6$	$6Q_1^5Q_2-20Q_1^3Q_2^3+6Q_1Q_2^5$
$n=7$	$Q_1^7-21Q_1^5Q_2^2+35Q_1^3Q_2^4-7Q_1Q_2^6$	$7Q_1^6Q_2-35Q_1^4Q_2^3+21Q_1^2Q_2^5-Q_2^7$
$n=8$	$Q_1^8-28Q_1^6Q_2^2+70Q_1^4Q_2^4-$ $-28Q_1^2Q_2^6+Q_2^8$	$8Q_1^7Q_2-56Q_1^5Q_2^3+56Q_1^3Q_2^5$- $-8Q_1Q_2^7$

with the new Hamiltonian

$$\mathcal{H}=\frac{P_1^2+P_2^2}{8(Q_1^2+Q_2^2)}+\frac{1}{2}(P_1Q_2-P_2Q_1)+\frac{q}{1+q}(Q_1^2-Q_2^2)-$$
$$-\frac{1}{1+q}\frac{1}{(Q_1^2+Q_2^2)}-\frac{q}{1+q}\cdot\frac{1}{\sqrt{(Q_1^2+Q_2^2-1)^2+4Q_1^2Q_2^2}}-\frac{q^2}{2(1+q)^2} \qquad (6)$$

Introducing the time transformation $\frac{dt}{d\tau}=r_1^3r_2^3$, where τ is the fictitious time, the motion of the system is slowed down, in order to observe and study the movement of the system around the singularity points (Szücs-Csillik *et al.*, 2012).

3. GENERALIZED LC – LCn

We denote $z=Q_1+iQ_2$ a single complex variable and $h:\Omega\to\mathbb{C}$, $h(z)=h(Q_1+iQ_2)=f(Q_1,Q_2)+ig(Q_1,Q_2)$ a complex-valued function, where f and g are two real functions depending on two real variables Q_1 and Q_2. If $h(z)$ is a complex function, then its real and imaginary parts are harmonic functions (Carathéodory, 2001). Considering $h(z)=z$ it results $h(z^n)=z^n$, $n\in\mathbb{N}$ and $z^n=(Q_1+iQ_2)^n$. We obtain so the harmonic polynomials presented in the table 1 (Roman *et al.*, 2014).

Let denote $f_n=\Re(z^n)$ and $g_n=\Im(z^n)$, $n\in\mathbb{N}$, $n\geq 2$. By consequence we obtain:

$$f_n^2+g_n^2=(Q_1^2+Q_2^2)^n,\ D_n=\left(\frac{\partial f_n}{\partial Q_1}\right)^2+\left(\frac{\partial f_n}{\partial Q_2}\right)^2=n^2(Q_1^2+Q_2^2)^{n-1}.$$

Observation

In the Levi-Civita case ($n=2$) we obtain

$$f_2=Q_1^2-Q_2^2,\ g_2=2Q_1Q_2,\ f_2^2+g_2^2=(Q_1^2+Q_2)^2,\ D_2=4(Q_1^2+Q_2^2).$$

The equation of Hamiltonian becomes in the generalized LC (LCn) case

$$\begin{aligned} \mathcal{H}(Q_1,Q_2,P_1,P_2) &= \frac{1}{2D_n}(P_1^2+P_2^2)+\frac{P_1Q_2-P_2Q_1}{n}+\frac{q}{1+q}f_n- \\ &- \frac{1}{1+q}\cdot\frac{1}{\overline{r}_{1n}}-\frac{q}{1+q}\cdot\frac{1}{\overline{r}_{2n}}-\frac{q^2}{2(1+q)^2} \end{aligned} \tag{7}$$

and the canonical equations of motion will be

$$\begin{aligned} \frac{\mathrm{d}Q_1}{\mathrm{d}t} &= \frac{P_1}{D_n}+\frac{Q_2}{n} \\ \frac{\mathrm{d}Q_2}{\mathrm{d}t} &= \frac{P_2}{D_n}-\frac{Q_1}{n} \\ \frac{\mathrm{d}P_1}{\mathrm{d}t} &= \frac{(n-1)Q_1(P_1^2+P_2^2)}{D_n(Q_1^2+Q_2^2)}+\frac{P_2}{n}-\frac{q}{1+q}\frac{\partial f_n}{\partial Q_1}+ \\ &+ \frac{1}{(1+q)}\frac{\partial}{\partial Q_1}\left(\frac{1}{\overline{r}_{1n}}\right)+\frac{q}{(1+q)}\frac{\partial}{\partial Q_1}\left(\frac{1}{\overline{r}_{2n}}\right) \\ \frac{\mathrm{d}P_2}{\mathrm{d}t} &= \frac{(n-1)Q_2(P_1^2+P_2^2)}{D_n(Q_1^2+Q_2^2)}-\frac{P_1}{n}-\frac{q}{1+q}\frac{\partial f_n}{\partial Q_2}+ \\ &+ \frac{1}{(1+q)}\frac{\partial}{\partial Q_2}\left(\frac{1}{\overline{r}_{1n}}\right)+\frac{q}{(1+q)}\frac{\partial}{\partial Q_2}\left(\frac{1}{\overline{r}_{2n}}\right) \end{aligned} \tag{8}$$

where $\overline{r}_{1n}=\sqrt{f_n^2+g_n^2}$ and $\overline{r}_{2n}=\sqrt{(f_n-1)^2+g_n^2}$.

Observation

In order to solve the canonical equations of motion, we introduce the fictitious time τ, and make the time transformation $\frac{\mathrm{d}t}{\mathrm{d}\tau}=\overline{r}_{1n}^{n}\overline{r}_{2n}^{3}$. Introducing a time transformation singular equations of motion become regular equations of motion.

4. LEO NUMERICAL APPLICATION

Low Earth Orbit (LEO) is generally defined as an orbit between 160 kilometers (with a period of about 88 minutes) and 2000 kilometers (with a period of about 127 minutes) above the Earths surface. While a majority of artificial satellites are placed in LEO, making one complete revolution around the Earth in about 90 minutes, many communication satellites require geo-stationary orbits, and move at the same angular velocity as the Earth. As an application, we study the Iridium 18 (24872) LEO satellite, which is one of the Iridium satellite constellation (a large group of satellites providing voice and data coverage to satellite phones, pagers and integrated transceivers over Earth's entire surface). From NORAD web page we get the orbital elements of

Iridium 18, and we calculated the position vector (q_1, q_2, q_3) = (4.92, 5.19, - 0.01), and the velocity vector (p_1, p_2, p_3) = (- 0.02, 0.01, - 0.44). In order to obtain trajectories, the canonical equations of motion of the test particle must be integrated using initial conditions (we investigate the motion of the satellite in plane). We denote

$$q_{10} = q_1(t)/_{t=0}\ ,\ \ q_{20} = q_2(t)/_{t=0}\ ,\ \ p_{10} = p_1(t)/_{t=0}\ ,\ \ p_{20} = p_2(t)/_{t=0}$$

the initial conditions for the canonical equations in the physical plane. And

$$Q_{10} = Q_1(t)/_{t=0}\ ,\ \ Q_{20} = Q_2(t)/_{t=0}\ ,\ \ P_{10} = P_1(t)/_{t=0}\ ,\ \ P_{20} = P_2(t)/_{t=0}$$

denote the initial conditions for the canonical equations in the regularized plane. The connection between these initial conditions is given by the equations:

$$\begin{aligned} q_{10} &= f(Q_{10}, Q_{20}) \\ q_{20} &= g(Q_{10}, Q_{20}) \\ P_{10} &= p_{10}\left(\frac{\partial f}{\partial Q_1}\right)_{(Q_{10},\,Q_{20})} + p_{20}\left(\frac{\partial g}{\partial Q_1}\right)_{(Q_{10},\,Q_{20})} \\ P_{20} &= -p_{10}\left(\frac{\partial g}{\partial Q_1}\right)_{(Q_{10},\,Q_{20})} + p_{20}\left(\frac{\partial f}{\partial Q_1}\right)_{(Q_{10},\,Q_{20})} \end{aligned} \tag{9}$$

Initial condition in physical plane of the Iridium 18 is given in following: $q_{10} = 4.921$, $q_{20} = 5.194$, $p_{10} = -0.021$, $p_{20} = 0.018$. Initial condition calculated in LC regularized plane is: $Q_{10} = 2.457$, $Q_{20} = 1.056$, $P_{10} = -0.065$, $P_{20} = 0.13$. Initial condition computed in $LC3$ regularized plane is: $Q_{10} = 1.856$, $Q_{20} = 0.515$, $P_{10} = -0.097$, $P_{20} = 0.2924$.

In the following we present two regularization method, one is the well-known Levi-Civita regularization for $n = 2$ (LC) and the second is the generalized Levi-Civita regularization of order 3, denoted with $LC3$. Using Levi-Civitas coordinate transformation (4) the equations of motion of the restricted three-body problem becomes (5). The generalized $LC3$ ($n = 3$) coordinate transformation

$$q_1 = Q_1^3 - 3Q_1Q_2^2,\ q_2 = 3Q_1^2Q_2 - Q_2^3 \tag{10}$$

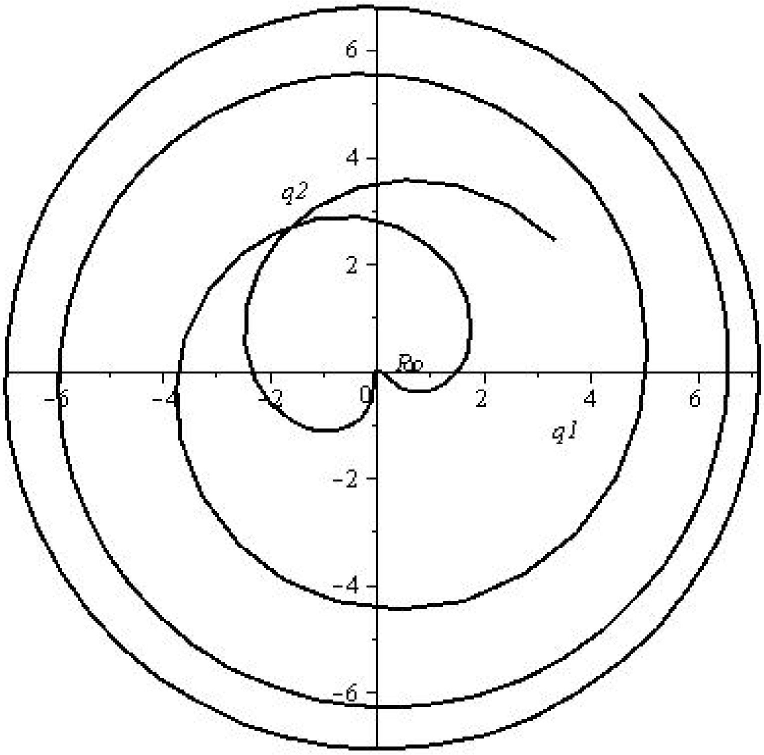

Fig. 1 – Motion in physical plane.

transforms the equations of motion, and we obtain the following canonical equations:

$$\begin{aligned}
\frac{\mathrm{d}Q_1}{\mathrm{d}t} &= \frac{P_1}{D}+\frac{Q_2}{3} \\
\frac{\mathrm{d}Q_2}{\mathrm{d}t} &= \frac{P_2}{D}-\frac{Q_1}{3} \\
\frac{\mathrm{d}P_1}{\mathrm{d}t} &= \frac{P_2}{3}+\frac{2Q_1(P_1^2+P_2^2)}{9(Q_1^2+Q_2^2)^3}-\frac{3q(Q_1^2-Q_2^2)}{1+q}-\frac{3}{1+q}\frac{Q_1}{R_1(Q_1^2+Q_2^2)}- \\
&- \frac{3q}{1+q}\frac{Q_2(Q_1^4+2Q_1^2Q_2^2+2Q_1+Q_2^4)}{R_2^3} \\
\frac{\mathrm{d}P_2}{\mathrm{d}t} &= -\frac{P_1}{3}+\frac{2Q_2(P_1^2+P_2^2)}{9(Q_1^2+Q_2^2)^3}-\frac{6qQ_1Q_2}{1+q}-\frac{3}{1+q}\frac{Q_2}{R_1(Q_1^2+Q_2^2)}- \\
&- \frac{3q}{1+q}\frac{Q_1^5+2Q_1^3Q_2^2+Q_1Q_2^4-Q_1^2+Q_2^2}{R_2^3}
\end{aligned} \tag{11}$$

where

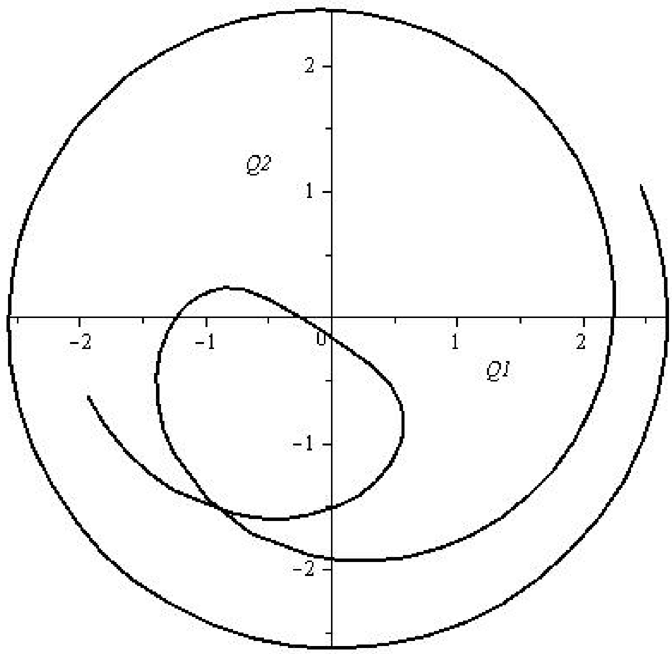

Fig. 2 – Motion in LC regularized plane.

$$
\begin{aligned}
D &= 9(Q_1^2+Q_2^2)^2,\\
R_1 &= (Q_1^2+Q_2^2)^{3/2},\\
R_2 &= \sqrt{Q_1^6+3Q_1^4Q_2^2-2Q_1^3+3Q_1^2Q_2^4+6Q_1Q_2^2+1+Q_2^6},
\end{aligned}
$$

Comparing LC and $LC3$ regularization, we analyze the slowing down behavior and the blowing up effect. In Fig. 1 we can see the motion in physical plane with a close approach point at $(0,0)$. In Fig. 2 and Fig. 3 the motion is in regularized plane without singularity. Comparing Figures 2 and 3 we can see the characteristic of the used regularization methods, namely LC and $LC3$, the scale diminution and the motion's blow up effect. This shows the importance of the regularization study close to the Earth.

5. CONCLUSION

Applying LCn for $n=2$ the Levi-Civita regularization methods is found. From theoretical point of view it is interesting to encapsulate this method in a family of methods which all conserve the Levi-Civita method properties. We integrated these equations, using initial conditions, obtained from the initial conditions used in the

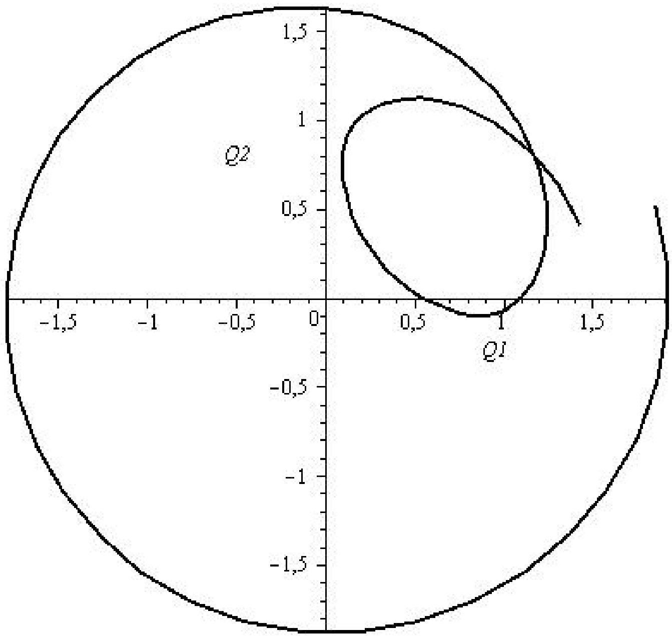

Fig. 3 – Motion in $LC3$ regularized plane.

physical plane. Using the regularization we realize: that the new canonical equations of motion are without singularity (regular), so the numerical integrator is faster, the trajectories conserve the shapes of the orbit (near the collision the manifold blow up), the motion is slowed down. In the case of LEO Iridium 18 satellite, the regularization study is necessary, because the lifetime of the spacecraft depends on the modelling errors.

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