A REVIEW OF THE PLANAR CALEDONIAN FOUR-BODY PROBLEM

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Abstract. The Caledonian four-body problem was introduced by Steves and Roy (1998). The Caledonian problem is shown to be suitable for application to the study of real four-body stellar systems. The coplanar CSFBP involves two pairs of equals or distinct masses moving in coplanar, initially circular orbits, starting in a collinear arrangement. In this paper we make a review of the Caledonian four-body problem, in particular the case of two pairs of equal masses and the evolution of equilibrium points.

Key words: four-body problem, Caledonian problem, equilibrium points..

1. INTRODUCTION

The motion of systems of n-bodies under their mutual gravitational attraction has always fascinated mathematicians and astronomers. Today the few-body problem is recognized as a standard tool in astronomy and astrophysics, from solar system dynamics to galactic dynamics Murray *et al.* (1999). Approximately two thirds of the stars in our Galaxy exist as part of multistellar systems. Stellar situations in the solar system can be seen as living in the province of few body problems, where computations of the orbits of these systems must be done in a precise way. In the same direction as several restricted three body problems have given much insight about real three body problems, the study of special type of four-body problem can be of help to understand the dynamical behavior of quadruple stellar systems through analytical or numerical studies. In the case of four-body problems, even the case of a restricted one can impose greater difficulties from an analytical point of view.

The number of quadruple stellar systems in the Galaxy is estimated to be of the order of thousands of millions. It is worth to note that the four-body problem is increasingly being used for explaining many complex dynamical phenomena that appear in the solar system and exoplanetary systems.

A very restricted four-body problem has been used to study a Sun-Earth-Moon-Satellite system to find possible regions of motion, also four-body problems involving three bodies of small mass revolving around a more massive body in the same plane have been used to explore the stabilizing role of Saturn in the evolution of the

Romanian Astron. J. , Vol. **24**, No. 1, p. 61–73, Bucharest, 2014

Sun-Jupiter-Saturn-asteroid system (Széll *et al.*, 2004).

The goal of this paper is to give a review of a family of coplanar four-body models for both linear and double-binary hierarchical systems known as coplanar Caledonian four-body problem (CFBP, for short), developed by Steves and Roy in Roy and Steves (1998), where four masses move in coplanar orbits about the center of mass of the system, initially in circular orbits. At $t = 0$ the bodies form a collinear configuration and move co-rotationally with velocity vectors perpendicular to that line and are by definition coplanar, see Figure 1. One interesting dynamical feature of this problem is that several known and already studied four-body problems like the rhomboidal, trapezoidal, equilateral, and collinear problems can be seen as subsystems.

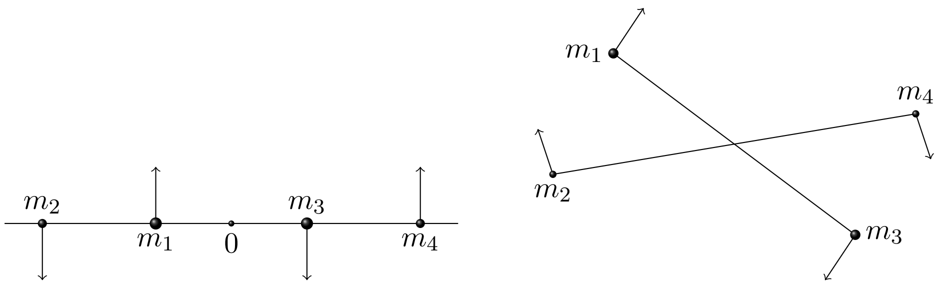

Fig. 1 – The CSFBP configuration for $t = 0$ and $t > 0$.

The CFBP enables considerable simplification to be made particularly in the form of the so called Caledonian Symmetric Double Binary Problem (CSDBP, for short). In the CSFBP two pairs of bodies of equal masses move in a fixed plane, occupying positions of central symmetry with respect to the origin. This system has four degrees of freedom like the planar three-body problem, see Steves and Roy (2001). There are two types of symmetrical restrictions: a past-future symmetry so that the dynamical evolution of the system after time $t = 0$ is a mirror image of that before $t = 0$; and a dynamical symmetry for which the dynamical evolution of two bodies on one side of the system's center of mass is a rotational image of that for the two bodies on the other side of the center of mass.

The CSFBP as proposed by Steves and Roy is relevant in studying the stability and evolution of symmetric quadruple stellar clusters and exoplanetary systems of two planets orbiting a binary pair of stars, see for example Steves and Roy (2001) and Széll *et al.* (2004). Besides, by using the CSFBP Steves and Roy exploited both past-future and dynamical symmetries and reduced the number of variables of the system to study the dynamical behavior of one of the binary pairs, the other binary pair's motion being a mirror image of the first binary pair's motion.

In the solar system, orbital motion in quadruple stellar systems is invariable found to consist of perturbed two-body motions. All four-body stellar systems are found to exist in two different hierarchies, the double binary and the linear. According to Roy and Steves (1999), a dynamical system of n bodies is said to be hierarchical if at a given epoch it can be defined to exist as a clearly identified number of disturbed two-body motions, where the two bodies in a pair may be made up of masses, a mass and a center of mass, or two centres of mass.

The hierarchy is said to be changed if any of the designated disturbed two-body systems is disrupted. There are four different types of hierarchy states present in the CSFBP two double binaries (m_1-m_2 and m_3-m_4) and two single binaries (m_1-m_3 and m_2-m_4). Also four different hierarchical arrangements of the four bodies which are shown in Figure 2, and these arrangements are listed next.

According to Steves and Roy (1999), the CSFBP can be found in only one of four different hierarchy states:

- The bodies m_1-m_2 and m_3-m_4 form binaries which orbit around the barycenter of the four-body system, Figure 2 (a).

- The bodies m_1-m_3 and m_2-m_4 form binaries which orbit around the barycenter of the four-body system, Figure 2 (b).

- The bodies m_2-m_3 form a central binary and the and m_1-m_4 bodies orbit around it, Figure 2 (c).

- The bodies m_1-m_4 form a central binary and the and m_2-m_3 bodies orbit around it, Figure 2 (d).

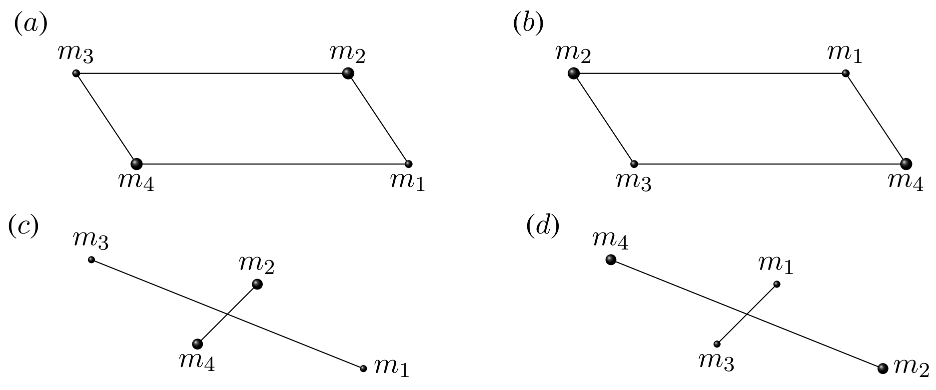

Fig. 2 – The four possible hierarchical arrangements in the CSFBP.

2. THE EQUATIONS OF MOTION

The equations of motion of the n-body problem of mass m_i are given by

$$m_i\ddot{\mathbf{r}}_i = \sum_{i\neq j}\frac{m_i m_j}{r_{ij}^3}\mathbf{r}_{ij}, \qquad i = 1,2,3,\ldots \tag{1}$$

where $\mathbf{r}_i = (x_i, y_i)$ correspond to the positions of the bodies, $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$ are the relative positions with respect each other, r_{ij} are the distances between the particles with positions $\mathbf{r}_i$ and $\mathbf{r}_j$ and units are chosen so that the gravitational constant is $G = 1$. For the full four-body problem the equations of motions become

$$\begin{aligned}
\ddot{\mathbf{r}}_1 &= \frac{m_2}{r_{12}^3}\mathbf{r}_{12} + \frac{m_3}{r_{13}^3}\mathbf{r}_{13} + \frac{m_4}{r_{14}}\mathbf{r}_{14},\\
\ddot{\mathbf{r}}_2 &= \frac{m_1}{r_{12}^3}\mathbf{r}_{21} + \frac{m_3}{r_{23}^3}\mathbf{r}_{23} + \frac{m_4}{r_{24}}\mathbf{r}_{24},\\
\ddot{\mathbf{r}}_3 &= \frac{m_1}{r_{31}}\mathbf{r}_{31} + \frac{m_2}{r_{32}^3}\mathbf{r}_{32} + \frac{m_4}{r_{34}}\mathbf{r}_{34},\\
\ddot{\mathbf{r}}_4 &= \frac{m_1}{r_{41}}\mathbf{r}_{41} + \frac{m_2}{r_{42}}\mathbf{r}_{42} + \frac{m_4}{r_{43}}\mathbf{r}_{43}.
\end{aligned}$$

3. CALEDONIAN PROBLEM WHERE THE MASSES ARE TAKEN PAIRWISE EQUAL

In the symmetrical case we are confined to the model where there are two pairs of bodies with the members of each pair symmetrically linked in mass and dynamics. For this symmetrical problem we also consider the particular case when the masses of the four bodies are equal. In this section we will follow closely Roy and Steves (1998). The highly symmetric cases of the Caledonian problem of four equal masses correspond to arrangements in squares (A), equilateral triangles (B) and collinear (C) configurations.

- Square (A)

  $$\mathbf{r}_4 = -\mathbf{r}_2, \qquad \mathbf{r}_3 = -\mathbf{r}_1, \qquad \mathbf{r}_{34} = -\mathbf{r}_{12}, \qquad \mathbf{r}_{41} = -\mathbf{r}_{23}$$

  are the relations among the positions of the bodies.

- Equilateral triangle (B)
  In this case we have $\mathbf{r}_4 = \mathbf{0}$ and the relations among the distances of the bodies are given a follows $r_{12} = r_{23} = r_{31} = a$ and $r_1 = r_2 = r_3 = a/\sqrt{3}$.

- Collinear configuration (C)
  In order to keep the symmetry when evolving the bodies, their positions are

given by the equations

$$\mathbf{r}_4 = -\mathbf{r}_1, \qquad \mathbf{r}_3 = -\mathbf{r}_2.$$

Let the mass ratio $\mu = m/M \leq 1$. We now let two of the bodies in cases A, B and C where one pair of bodies each has a mass equal $m = \mu M$, where M is the mass of each of the other pair of bodies.

The possible cases for $0 < \mu < 1$ are shown diagrammatically in Figure 3. We note that only in cases (h) and (g) does no symmetry exist for $\mu \neq 1$. In the other cases, use can be made of the symmetry existing to reduce the number of variables. In all cases value of the mass ratio μ, the initial geometry remains unaltered in shape or size but the system rotates with constant angular velocity about the centre of mass.

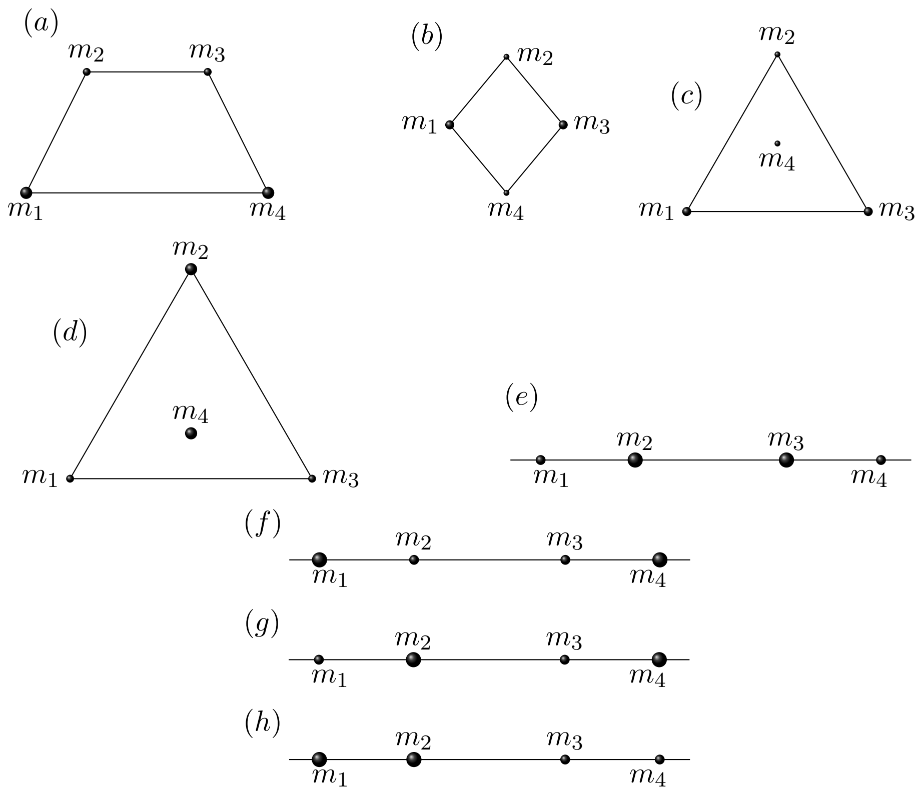

Fig. 3 – (a) Trapezoidal configuration: $m_1 = m_4 = M$, $m_2 = m_3 = m$. (b) Rhomboidal configuration: $m_1 = m_3 = M$, $m_2 = m_4 = m$. (c) Equilateral triangle: $m_1 = m_3 = M$, $m_2 = m_4 = m$. (d) Equilateral triangle: $m_1 = m_3 = m$, $m_2 = m_4 = M$. Collinear configurations: (e) $m_1 = m_4 = m$, $m_2 = m_3 = M$; (f) $m_1 = m_4 = M$, $m_2 = m_3 = m$; (g) $m_1 = m_3 = m$, $m_2 = m_4 = M$; (h) $m_1 = m_2 = M$, $m_3 = m_4 = m$.

Let us remark that there are two limit problems, namely $\mu = 1$ case of four equal masses, and $\mu = 0$ correspond to Lagrange solution of the Copenhagen problem.

3.1. SUBPROBLEMS

There are several subproblems of the CSFBP that have been studied by several authors, as listed below.

- *Rhomboidal problem*
  The rhomboidal four-body problem means a configuration of four particles with masses m_1, m_2, m_3, m_4 respectively, that are located in the plane at the vertices of a rhombus, where $m_1 = m_3 > 0$ are in the horizontal direction and $m_2 = m_4 > 0$ in the vertical direction. The particles are given symmetric initial conditions in positions and velocities with respect to the axes in the plane and always kept a symmetric rhomboidal configuration under the law of Newton attraction. This problem has been studied, among others, by Lacomba and Perez (1993) and Ji *et al.* (2000).

- *Trapezoidal problem*
  The trapezoidal four-body problem is a planar problem, where symmetry assumptions are imposed upon the values of the masses $m_1 = m_4$ and $m_2 = m_3$ and initial conditions so that the resulting solution defines a symmetrical trapezoid for all time. This problem has three degrees of freedom and has been studied, among others, by Lacomba (1981) and Simó and Lacomba (1982).

- *Collinear problem*
  Collinear symmetric four bodies the motion is restricted to a line where the inner pair have mass $m_2 = m_3$ and the outer pair have mass $m_1 = m_4$. This is a two degrees of freedom problem and has been studied, among others, by Lacomba and Medina (2004) and Ouyang and Xie (2009).

- *Equilateral four-body problem.*
  To the best of the knowledge of the authors, this problem has not been studied to this date.

4. EQUILIBRIUM POINTS

The Copenhagen problem is the restricted circular coplanar three-body problem with two bodies of equal finite mass revolving in a circle about their common center of mass with a third particle of infinitesimal mass moving in an orbit coplanar with that of the two finite masses. This problem was one of the main subjects of E.

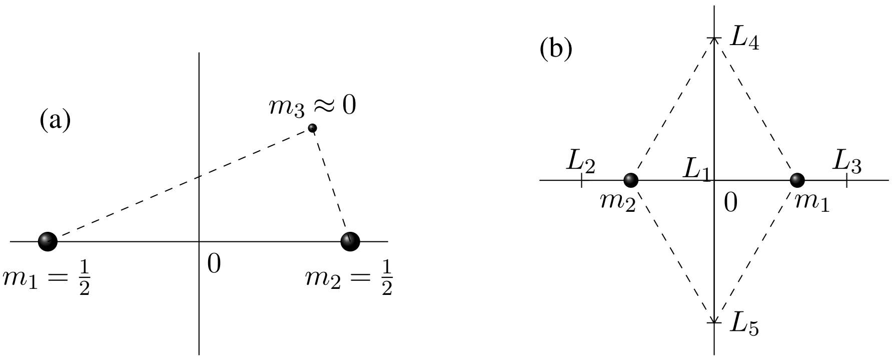

Fig. 4 – (a) The Copenhagen problem in the rotating set where masses m_1 and m_2 are located at $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$, respectively. (b) Lagrange points in Copenhagen problem located at $(0,0)$, $(-1.19841,0)$, $(1.19841,0)$, $(0,0.866)$ and $(0,-0.866)$.

Strömgren and the Copenhagen school during the beginning of the twentieth century and they computed a great number of periodic orbits, see Szebehely (1967).

It is known that an equilibrium configuration of four bodies is a geometric configuration of four bodies in which the gravitational forces are balanced in such a way that the four bodies rotate together about their centre of mass and thus the geometric configuration is maintained for all time. In this section we present a detailed summary of the equilibrium configurations given by Roy and Steves (1998).

As we know, the Lagrange solutions of the three-body problem are given by five Lagrange points of equilibrium, three of which, L_1, L_2 and L_3, remain in line with the two finite masses. The fourth and fifth Lagrange points of equilibrium, L_4 and L_5 form equilateral triangles with m_1 and m_2, see Figure 4.

The dynamical relation of the Copenhagen three-body problem and the Caledonian four-body problem is more subtle and elegant that it seems to be, as we see next. Roy and Steves (1998) showed that by taking two pairs of equal masses in the planar Caledonian four-body, with two of the bodies having mass equal to m, each of the remaining two bodies with mass M and considering the relation between the values m and M given by $m = \mu M$, the evolution of the families of equilibrium solutions for all values of μ, when μ approaches to zero is as follows, these solutions reduce to the Lagrange solutions L_1, L_2, L_3, L_4 and L_5 of the Copenhagen problem when two of the masses are equally reduced, see Figure 5.

Summing-up, Roy and Steves (1998) showed the evolution of the equilibrium points for $0 < \mu < 1$, beginning at the four-body equals mass solution of the square where $\mu = 1$ and culminating in the L_4 and L_5 solutions where μ approaches zero, see left picture in Figure 5. Besides, the right picture in Figure 5 shows the equivalent families of solutions as μ ranges from one to zero for the other cases. The large bodies in the pictures indicate the locations of the two large masses which are equal to M and the locations of the other two bodies when their masses are also equal to M (that is, $\mu = 1$). The smaller bodies indicate the locations of the other pair of bodies as μ reduces to zero. Where even smaller bodies are depicted, these show the locations of reduced masses when μ varies from 0.1 to 0.

4.1. NON COLLINEAR EQUILIBRIUM POINTS

In this section we shall discuss on the non collinear equilibrium configurations for the four-body problem consisting of two pairs of equal masses bodies when the parameter of mass μ varies from 1 to 0. There are four cases, two of them beginning at square configurations but presenting two different kinds of evolution and the two other cases having triangular configurations as initial configurations with two different evolution stories. In the first of the next items we present the first two cases, those where the initial configurations are squares while the case of triangular initial configurations are considered in the second ítem.

i) Let us start by watching at the left picture of Figure 5 where the solution begins as a square for $\mu = 1$, where all the masses are equal; then, while μ approaches zero, the configuration changes to a trapezoid and finally ends as the Lagrange equilibrium triangular solution when $\mu = 0$, reaching Lagrange point L_4. For the right picture in the same Figure 5, we start at $\mu = 1$with a square solution and the evolution of the configuration when μ goes to zero gives a continuous family of solutions that reaches the equilateral triangular solution of the Copenhagen problem. The evolution and final Lagrange solution points for the two small bodies as their masses are reduced to zero has still to be explored. The two small masses could migrate to the L_4 or L_1, points or both.

ii) Now, we consider the triangular equilibrium configuration of four bodies with two pairs of equal masses, where two large masses M lie at the two vertices of a triangle and two smaller masses m lie at on the line of symmetry of the triangle. In Shoaib (2007) showed numerically, that there are two families of equilibrium configurations for each value of μ varying from 1 to zero, and these are shown in Figure 6. In this figure the origin the point halfway between the two primaries of masses M. Thus unlike in Figure 5, the centre of mass is a point that moves as μ is reduced from 1 to 0. For this first case of triangular equilibrium configurations, two special solutions give the boundary of the family of solutions. For $\mu = 1$ (four equal masses case) there are two well known solutions, the isosceles triangle solution and the equilateral triangle solution. So, there are solutions that start with an isosceles triangle configuration when $\mu = 1$ and solutions that start with a equilateral triangle configuration for the same value of μ. The other boundary of the families of solutions correspond to the value $\mu = 0$. For this

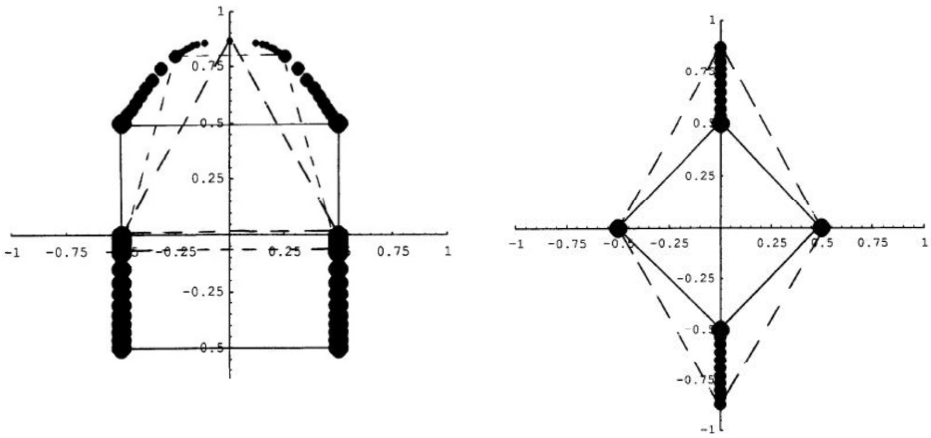

Fig. 5 – Non collinear families of equilibrium solutions. Left picture shows the change in the equilibrium solution as μ is varied from 1 to 0, beginning at the four-body equal mass solution of square where $\mu = 1$ and culminating in the L_4 and L_5 solutions where μ approaches zero. Right picture shows the evolution of equivalent families of solutions as μ is varied from 1 to 0 for the other cases. These figures were taken from Roy and Steves (1998).

null value of μ, we obtain the Lagrange solutions of the Copenhagen problem. When starting with an isosceles triangular four-body configuration, as μ goes from 1 to 0, the bodies on the line of symmetry of the triangle go to L_4, while the solution starting with an equilateral triangle four-body configuration behaves as follows, the upper point of the triangular four-body configuration goes to L_4 and the point in the interior of the triangle goes to L_1. In short, between $\mu = 1$ and $\mu = 0$ there are two families of equilibrium solutions.

4.2. COLLINEAR EQUILIBRIUM POINTS

In this section we consider all possible arrangements of the two pairs of equal masses along a straight line. The first two of the four arrangements are symmetric while the last two arrangements are nonsymmetric.

Let us consider four bodies in a collinear configuration having equal mass by pairs. Observe that a particular case is obtained when all of the particles of the gravitational system have equal mass. The possible arrangements for the four bodies when they are taken by pairs of equal mass are depicted in Figure 3 (e)-(h).

Next we review the four kinds of collinear equilibrium configurations both symmetric and non-symmetric.

i) The symmetrical arrangement of two pairs of different masses, the larger masses $(m_2 = m_3 = M)$ lie in the middle of the line and the smaller masses $(m_1 = m_4 =$

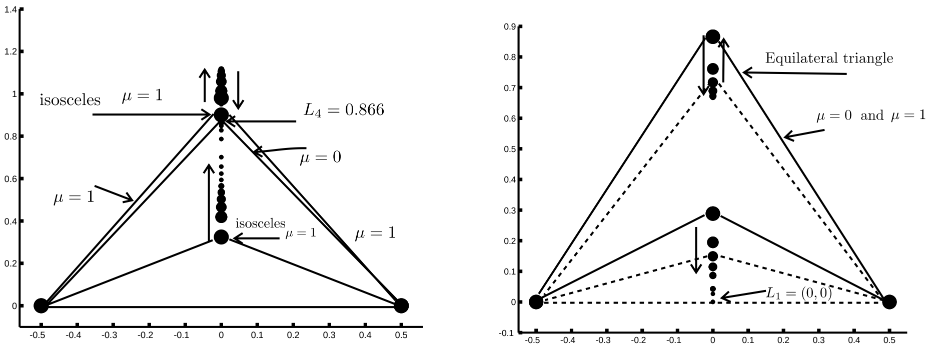

Fig. 6 – The evolution of all four masses when μ is varied from 1 to 0 in the triangular equilibrium case (left picture). Isosceles triangle (b) Solution equilateral triangle. Here the origin is located halfway between the two primaries and thus the centre of mass moves as μ is varied. These figures were taken from Shoaib (2007)

m) lie at the corners apiece, also placed symmetrically with respect to the center of mass of the four-body configuration as shown in the Figure 3 (e).

The Figure 7 shows the locations of the family of the solutions from the collinear equal mass solution at $\mu = 1$ to the solution at $\mu = 0$, where m_1 and m_4 have migrated to the L_3 and L_2 points, respectively, as their masses have been reduced to zero.

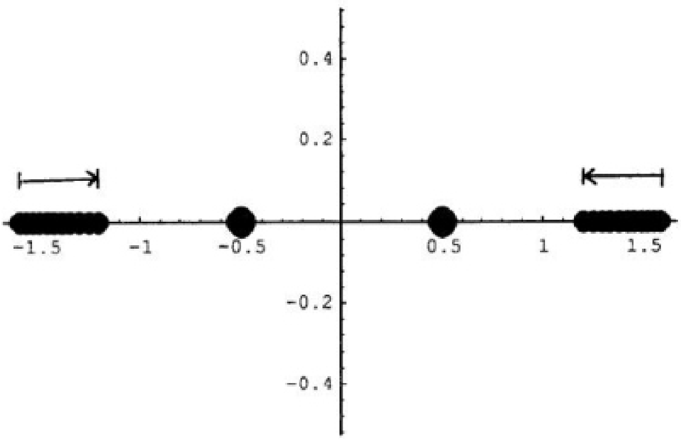

Fig. 7 – Collinear families of equilibrium solutions. m_1 and m_4 approach L_3 and L_2. This figure was taken from Roy and Steves (1998).

ii) In the second case, the pair of bodies with smaller mass ($m_2 = m_3 = m$) have symmetric positions with respect to the centre of mass and remain closer to it than the bodies with equal and larger masses ($m_1 = m_4 = M$), see Figure 3 (f).

When μ tends to 0 the bodies m_2 and m_3 approach L_1. It is shown in Figure 8.

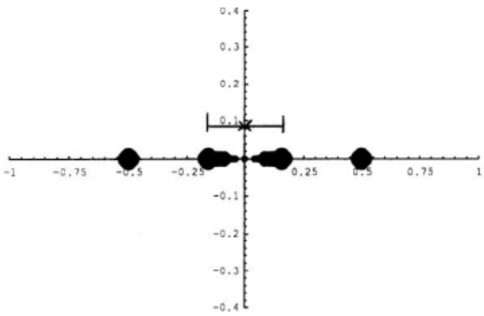

Fig. 8 – Collinear families of equilibrium solutions. m_2 and m_3 migrate to the L_1 when $\mu \to 0$. This figure was taken from Roy and Steves (1998).

iii) This case corresponds to a non symmetric configuration of the four-body collinear configuration. The masses are arranged in the following way: $m_1 = m$, $m_2 = M$, $m_3 = m$ and $m_4 = M$, see Figure 3 (g).

In this case, when the masses of m_1 and m_3 are reduced to zero they migrate to the L_3 and L_1 Lagrange point respectively. The fourth m_4 of mass M migrates to the point $1/2$ with m_1 remaining at $1/2$. It is shown in Figure 9.

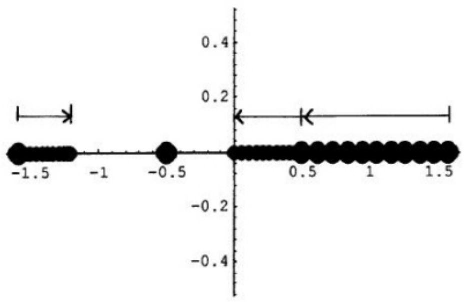

Fig. 9 – Collinear families of equilibrium solutions. m_3 migrates to L_1, while the other small body m_1 migrates to L_3 as their masses are reduced to zero. The body m_4 migrates to the point $1/2$ with m_1 remaining at $-1/2$. This figure is taken from Roy and Steves (1998).

iv) This is the second non-symmetric arrangement of the four bodies. The masses are arranged in the following way: $m_1 = M$, $m_2 = M$, $m_3 = m$ and $m_4 = m$, see Figure 3 (h).

For this configuration, the two smaller bodies $m_3 = m_4$ migrate to the L_2 Lagrange point when μ tends to 0. It may be noted that m_4 actually moves outer further than L_2 before returning to end at that point. m_2 migrates to the point $1/2$ as μ approaches zero. It is shown in Figure 10.

Note that for $\mu = 1$ we get the collinear equal mass solution.

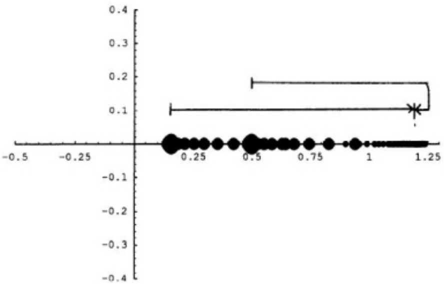

Fig. 10 – Collinear families of equilibrium solutions. m_4 moves out further to L_2 before returning to end at that point. m_2 migrates to the point $1/2$ as μ approaches zero. This figure was taken from Roy and Steves (1998).

In 2011, Shoaib and Faye (2011) also discussed about the equilibrium solutions of four different types of collinear four-body problems having two pairs of equal masses, they obtained similar results to those by Roy and Steves (1998). Their approach consisted in using two parameter of masses $\mu_1 = m_1/M_T$ and $\mu_2 = m_2/M_T$, where M_T is the total mass of the system, and m_1 and m_2 are the two unequal masses.

5. CONCLUSIONS

In this paper we have made a review of the planar caledonian four-body problem and special attention was paid to the Caledonian symmetric double binary problem having bodies with equal masses by pairs. We looked at the existence of a family of hierarchical subsystems for the caledonian problem and analyzed the behavior of the equilibrium solutions for the case of two pairs of bodies with equal mass each for values of the parameter $\mu = M/m$, where M is the mass of the pair of heavier bodies and m is the mass of pair of lighter bodies. This study permits us to see the more profound relation between a three body problem, the Copenhagen problem and a four-body problem, the double binary with equal masses by pairs. As μ approaches zero, four-body equilibrium solutions of the Caledonian symmetric double binary problem have as limit the Lagrange equilibrium points L_1, L_2, L_3, L_4 and L_5 for the

Copenhagen three body problem. It is interesting to point out that some subsystems of the plane Caledonian four-body problem, like the rhomboidal and the collinear symmetric four-body problems among others, have been studied by several authors and it is worth to consider for further study the relation of these problems seen as subsystems of the caledonian problem in order to have a deeper understanding of the four-body problem.

Acknowledgements. M. Alvarez–Ramírez has been partially supported by Red de cuerpos académicos Ecuaciones Diferenciales, Proyecto sistemas dinámicos y estabilización. PROMEP 2011-SEP México.

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Received on 3 April 2014