PERIODIC ORBITS IN THE ZERO–HOPF BIFURCATION OF THE RÖSSLER SYSTEM

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To the memory of Vasile Mioc

Abstract. A zero-Hopf equilibrium is an isolated equilibrium point whose eigenvalues are $\pm\omega i \neq 0$ and 0. For a such equilibrium there is no a general theory for knowing when from this equilibrium bifurcates a small–amplitude periodic orbit moving the parameters of the system. We provide here an algorithm for solving this problem. In particular, first we characterize the values of the parameters for which a zero–Hopf equilibrium point takes place in the Rössler systems, and we find two one–parameter families exhibiting such equilibria. After for one of these families we prove the existence of one periodic orbit bifurcating from the zero–Hopf equilibrium. The algorithm developed for studying the zero–Hopf bifurcation of the Rössler systems can be applied to other differential system in $\mathbb{R}^n$.

Key words: Rössler system, periodic orbit, averaging theory, zero Hopf bifurcation.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULT

In Rossler (1979) Rössler, inspired by the geometry of 3-dimensional flows, introduced several systems as prototypes of the simplest autonomous differential equations having chaos, the simplicity is in the sense of minimal dimension, minimal number of parameters and minimal nonlinearities. In MathSciNet appear at this moment more than 171 articles about the Rössler's systems.

Rössler invented a series of systems, the most famous is probably

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} = -y - z, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= \dot{y} = x + ay, \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= \dot{z} = bx - cz + xz,
\end{aligned} \tag{1}$$

introduced in Rossler (1979), see also Gaspard (2005). While the Rössler systems were created for studying the existence of strange attractors in differential systems of dimension three, many authors have studied the periodic orbits of these systems depending on their three parameters a, b and c. A brief summary of the results on the periodic orbits of the Rössler systems is done in section 2. The integrability of those systems was studied in Lima and Llibre (2011), see also the references quoted there.

We remark that in many papers the Rössler system is written into the form

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b - cz + xz. \end{aligned} \tag{2}$$

The differential systems (1) and (2) are equivalent. Indeed, changing the parameter b of system (2) by the new parameter $b = (c^2 - d^2)/a$, the two equilibrium points of system (2) are

$$p_\pm = \left(\frac{c \pm d}{2}, \frac{-c \mp d}{2a}, \frac{c \pm d}{2a} \right).$$

Translating to the origin of coordinates the equilibrium p_+ system (2) becomes system (1), after renaming the coefficients of x and z in the equation $\dot{z}$.

A *zero–Hopf equilibrium* is an equilibrium point of a 3–dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues. Usually the *zero–Hopf bifurcation* is a two–parameter unfolding (or family) of a 3-dimensional autonomous differential system with a zero–Hopf equilibrium. The unfolding has an isolated equilibrium with a zero eigenvalue and a pair of purely imaginary eigenvalues if the two parameters take zero values, and the unfolding has different topological type of dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin. This zero–Hopf bifurcation has been studied by Guckenheimer, Han, Holmes, Kuznetsov, Marsden and Scheurle in Guckenheimer (1980); Guckenheimer and Holmes (1990); Han (1998); Kuznetsov (2004); Scheurle and Marsden (1984), and it has been shown that some complicated invariant sets of the unfolding could be bifurcated from the isolated zero–Hopf equilibrium under some conditions. Hence, in some cases zero–Hopf bifurcation could imply a local birth of "chaos" see for instance the articles of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, Scheurle and Marsden (cf. Baldoma and Seara (2006, 2008); Broer and Vegter (1984); Champneys and Kirk (2004); Scheurle and Marsden (1984)).

As far as we know nobody has studied the existence or non–existence of zero–Hopf equilibria and zero–Hopf bifurcations in the Rössler systems. This will be our objective to study the zero–Hopf bifurcations in the Rössler systems. We must mention that the method used for studying the zero–Hopf bifurcation can be applied to any differential system in $\mathbb{R}^3$.

In the next proposition we characterize when the equilibrium point localized at the origin of coordinates of the Rössler systems is a zero–Hopf equilibrium point.

Proposition 1 . *There are two one–parameter families of Rössler systems for which the origin of coordinates is a zero–Hopf equilibrium point. Namely:*

(i) $a = c \in (-\sqrt{2}, \sqrt{2})$ *and* $b = 1$*; and*

(ii) $a = c = 0$ *and* $b \in (-1, \infty)$.

Proposition 1 is proved in section 4.

Note that the two families of Proposition 1 intersect at the point $a = c = 0$ and $b = 1$.

Now we shall study when the Rössler systems having a zero–Hopf equilibrium point at the origin of coordinates have a zero–Hopf bifurcation producing some periodic orbit.

Theorem 2 . *Let* $(a,b,c) = (\bar{a} + \varepsilon\alpha, 1 + \varepsilon\beta, \bar{a} + \varepsilon\gamma)$ *be with* $\bar{a} \in (-\sqrt{2}, \sqrt{2}) \setminus \{0\}$ *and* ε *a sufficiently small parameter. If*

$$\left(-\alpha + a\left(1 - a^2\right)\beta + \gamma\right)\left(\left(a^2 - 1\right)\alpha + a\beta + \left(1 - a^2\right)\gamma\right) < 0 \tag{3}$$

and

$$\alpha + a\beta - \gamma \neq 0, \tag{4}$$

then the Rössler system (1) *has a zero–Hopf bifurcation at the equilibrium point localized at the origin of coordinates, and a periodic orbit borns at this equilibrium when* $\varepsilon = 0$*, and it exists for* $\varepsilon > 0$ *sufficiently small. Moreover, the stability or instability of this periodic orbit is given by the eigenvalues*

$$\frac{A \pm \sqrt{B}}{2a^2(2 - a^2)^{3/2}}, \tag{5}$$

where

$$\begin{aligned} A = \;& (2 - a^2)(\alpha - a\beta - \gamma), \\ B = \;& \left(3a^4 - 4\right)\alpha^2 + 2a\left(2a^6 - 3a^4 + 4\right)\alpha\beta - 2\left(3a^4 - 4\right)\alpha\gamma \\ & + a^2\left(3a^4 - 4\right)\beta^2 - 2a\left(2a^6 - 3a^4 + 4\right)\beta\gamma + \left(3a^4 - 4\right)\gamma^2. \end{aligned}$$

Theorem 3 . *Let* $(a,b,c) = (\varepsilon\alpha, \bar{b} + \varepsilon\beta, \varepsilon\gamma)$ *be with* $\bar{b} \in (-1, \infty)$ *and* ε *a sufficiently small parameter. Using the averaging theory of first order we cannot find periodic orbits bifurcating from the zero–Hopf equilibrium point localized at the origin of coordinates of the Rössler system* (1).

Theorem 2 and 3 are proved in section 4 using the averaging theory of first order.

If $a(ab - c) \neq 0$ then the Rössler system (1) has a second equilibrium point, namely

$$p = \left(\frac{ab - c}{a}, c - ab, \frac{c - ab}{a}\right).$$

If we translate this equilibrium point at the origin of coordinates the Rössler system (1) becomes

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= \frac{c}{a}x - abz + xz. \end{aligned} \tag{6}$$

So system (6) coincides with system (1), if we rename the coefficients c/a and ab of $\dot{z}$ by b and c respectively. In other words, doing this change in the parameters of the system we can obtain equivalent theorems to Theorem 2, and 3 for the equilibrium p.

It is important to remark that the tools used here for studying the zero–Hopf bifurcations of the Rössler system can be applied for studying arbitrary zero–Hopf bifurcations of differential systems in $\mathbb{R}^n$ with $n > 2$.

2. RESULTS ON THE PERIODIC ORBITS

In this section we present a brief summary on the results about the periodic orbits of the Rössler system (1).

In Gledinning and Sparrow (1984), Glendinning and Sparrow studied the dynamics near homoclinic orbits, and applied their studies to the Rössler system showing the existence of periodic orbits near some homoclinic orbits.

Magnitskii (1995) did a qualitative analysis of the Hopf bifurcation (the appearance of periodic solutions) in the Rössler system. He finds the domain of the parameters a and b in which the Hopf bifurcation occurs as the bifurcation parameter c increases, obtains asymptotic formulas for the amplitude and period of the periodic solutions that arise in a neighborhood of the bifurcation point, and provide some information about their stability.

Krishchenko (1996) estimated domains for the existence of periodic orbits of differential systems in $\mathbb{R}^n$, and applied them to the Rössler system showing that all its periodic orbits lie inside a bounded domain.

Terekhin and Panfilova (1991) determined conditions for the existence of periodic solutions of the Rössler system in a neighborhood of the two equilibrium points of the system when $c^2 - 4ab > 0$.

Ashwin *et al.* (2003) studied numerically the dynamics of the Rössler system and provided some information on its periodic orbits.

Pilarczyk (2003) developed a numerical method based on the Conley index theory, and applied it to the Rössler system showing the existence of two periodic orbits.

Galias (2006) described a numerical method that approximates short–period orbits, he implemented and tested the method in the Rössler system.

Llibre *et al.* (2007) studied the Hopf bifurcation using averaging theory of first order, and applied it to the Rössler system.

Algaba *et al.* (2007) studied the periodic orbits and their bifurcations near a triple–zero singularity, and they apply their results to the Rössler system using numerical tools and they found an important number of bifurcations of periodic orbits.

Starkov and Starkov (2007) used compact invariant sets of the Rössler system for studying the existence and non–existence of periodic orbits.

3. THE AVERAGING THEORY FOR PERIODIC ORBITS

The averaging theory is a classical and matured tool for studying the behavior of the dynamics of nonlinear smooth dynamical systems, and in particular of their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace who provided an intuitive justification of the process. The first formalization of this procedure is due to Fatou (1928) in 1928. Important practical and theoretical contributions in this theory were made by Bogoliubov and Krylov (see Bogoliubov and Krylov (1934) and Bogoliubov (1945)). The averaging theory of first order for studying periodic orbits can be found in Verhulst (1991), see also Guckenheimer and Holmes (1990). It can be summarized as follows.

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst (1991).

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F(t,\mathbf{x}) + \varepsilon^2 G(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{7}$$

with $\mathbf{x} \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. Moreover we assume that both $F(t,\mathbf{x})$ and $G(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. We also consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{8}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(t,\mathbf{y})\mathrm{d}t. \tag{9}$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (7).

Theorem 4 . *Consider the two initial value problems (7) and (8). Suppose:*

(i) F, its Jacobian $\partial F/\partial x$, its Hessian $\partial^2 F/\partial x^2$, G and its Jacobian $\partial G/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty)\times D$

and $\varepsilon \in (0, \varepsilon_0]$.

(ii) F *and* G *are* $T-$*periodic in* t *(*T *independent of* ε*).*

Then the following statements hold.

(a) If p *is an equilibrium point of the averaged equation* (8) *and*

$$\det\left(\frac{\partial f}{\partial \mathbf{y}}\right)\bigg|_{\mathbf{y}=p} \neq 0, \tag{10}$$

then there exists a $T-$*periodic solution* $\varphi(t,\varepsilon)$ *of equation* (7) *such that* $\varphi(0,\varepsilon) \to p$ *as* $\varepsilon \to 0$.

(b) The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ *is given by the stability or instability of the equilibrium point* p *of the averaged system* (8)*. In fact the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\varphi(t,\varepsilon)$*.*

4. PROOFS

Proof 5 (Proof of Proposition 1) . *The characteristic polynomial of the linear part of the Rössler system at the origin is*

$$p(\lambda) = -\lambda^3 + (a-c)\lambda^2 + (ac-1-b)\lambda + ab - c.$$

Imposing that $p(\lambda) = -\lambda(\lambda^2+\omega^2)$*, we obtain*

(i) $a = c = \pm\sqrt{2-\omega^2}$ *and* $b = 1$ *for* $\omega \in (0,\sqrt{2})$*; and*

(ii) $a = c = 0$ *and* $b = \omega^2 - 1$ *for* $\omega \in (0,\infty)$.

So the proposition follows.

Proof 6 (Proof of Theorem 2) . *If* $(a,b,c) = (\bar{a}+\varepsilon\alpha, 1+\varepsilon\beta, \bar{a}+\varepsilon\gamma)$ *with* $\varepsilon > 0$ *a sufficiently small parameter, then the Rössler system becomes*

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + \big(\bar{a}+\varepsilon\alpha\big)y, \\ \dot{z} &= (1+\varepsilon\beta)x - \big(\bar{a}+\varepsilon\gamma\big)z + xz. \end{aligned} \tag{11}$$

Doing the rescaling of the variables $(x,y,z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$*, system* (11) *in the new variables* (X,Y,Z) *writes*

$$\begin{aligned} \dot{X} &= -Y - Z, \\ \dot{Y} &= X + \bar{a}Y + \varepsilon\alpha Y, \\ \dot{Z} &= X - \bar{a}Z + \varepsilon(\beta X - \gamma Z + XZ). \end{aligned} \tag{12}$$

Now we shall write the linear part at the origin of the differential system (12) *when* $\varepsilon = 0$ *into its real Jordan normal form, i.e. as*

$$\begin{pmatrix} 0 & -\sqrt{2-\bar{a}^2} & 0 \\ \sqrt{2-\bar{a}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For doing that we consider the linear change $(X,Y,Z) \to (u,v,w)$ *of variables given by*

$$\begin{aligned} X &= \frac{(\bar{a}^2-2)\,v - \bar{a}\left(\sqrt{2-\bar{a}^2}\,u + w\right)}{\bar{a}^2-2}, \\ Y &= \frac{\sqrt{2-\bar{a}^2}\,u + w}{\bar{a}^2-2}, \\ Z &= -\frac{\bar{a}(2-\bar{a}^2)v + \sqrt{2-\bar{a}^2}\left(\bar{a}^2-1\right)u + w}{\bar{a}^2-2}. \end{aligned} \tag{13}$$

In the new variables (u,v,w) *the differential system* (12) *writes*

$$\begin{aligned} \dot{u} = {} & -\sqrt{2-\bar{a}^2}\,v + \varepsilon \frac{1}{(2-\bar{a}^2)^{3/2}} \Big(\alpha\left(1-\bar{a}^2\right)\left(\sqrt{2-\bar{a}^2}u + w\right) \\ & + \left((\bar{a}^2-2)v - \bar{a}\left(\sqrt{2-\bar{a}^2}u + w\right)\right)\left(\beta + \frac{\left(\bar{a}^2-1\right)u}{\sqrt{2-\bar{a}^2}} + \bar{a}v - \frac{w}{\bar{a}^2-2}\right) \\ & + \gamma\left(\bar{a}(2-\bar{a}^2)v + \sqrt{2-\bar{a}^2}\left(\bar{a}^2-1\right)u + w\right)\Big), \\ \dot{v} = {} & \sqrt{2-\bar{a}^2}\,u + \varepsilon \frac{\alpha\bar{a}\left(\sqrt{2-\bar{a}^2}\,u + w\right)}{\bar{a}^2-2}, \\ \dot{w} = {} & \varepsilon \frac{1}{\bar{a}^2-2}\Big(\alpha\left(-\sqrt{2-\bar{a}^2}u - w\right) + \bar{a}\left((\bar{a}^2-2)v - \left(\sqrt{2-\bar{a}^2}u + w\right)\right) \\ & \left(\frac{\left(\bar{a}^2-1\right)u}{\sqrt{2-\bar{a}^2}} + \bar{a}v - \frac{w}{\bar{a}^2-2} + \beta\right) \\ & + \gamma\left(\bar{a}(2-\bar{a}^2)v + \sqrt{2-\bar{a}^2}\left(\bar{a}^2-1\right)u + w\right)\Big). \end{aligned} \tag{14}$$

Now we pass the differential system (14) *to cylindrical coordinates* (r,θ,w) *defined by* $u = r\cos\theta$ *and* $v = r\sin\theta$*, and we obtain*

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} = \; & \varepsilon\frac{1}{\sqrt{2-\bar{a}^2}\,r}\Big(\frac{\alpha\bar{a}\sqrt{2-\bar{a}^2}\,r^2\cos\theta\sin\theta}{\bar{a}^2-2}+\frac{\alpha\bar{a}rw\sin\theta}{\bar{a}^2-2}\\
&+\frac{r\cos\theta}{(2-\bar{a}^2)^{3/2}}\Big(\alpha\left(1-\bar{a}^2\right)\Big(w+\sqrt{2-\bar{a}^2}\,r\cos\theta\Big)\\
&+\Big(\beta-\frac{w}{\bar{a}^2-2}+\frac{(\bar{a}^2-1)r\cos\theta}{\sqrt{2-\bar{a}^2}}+\bar{a}r\sin\theta\Big)\\
&\Big(-\bar{a}w-\bar{a}\sqrt{2-\bar{a}^2}r\cos\theta+\left(\bar{a}^2-2\right)r\sin\theta\Big)\\
&+\gamma\big(w+\sqrt{2-\bar{a}^2}\left(\bar{a}^2-1\right)r\cos\theta-\bar{a}\left(\bar{a}^2-2\right)r\sin\theta\big)\Big)\Big)+O(\varepsilon^2)\\
= \; & \varepsilon F_1(\theta,r,w)+O(\varepsilon^2),\\
\frac{\mathrm{d}w}{\mathrm{d}\theta} = \; & \varepsilon\frac{1}{\sqrt{2-\bar{a}^2}\,(\bar{a}^2-2)}\Big(-\alpha\Big(w+\sqrt{2-\bar{a}^2}\,r\cos\theta\Big)\\
&+\Big(\beta-\frac{w}{\bar{a}^2-2}+\frac{\left(\bar{a}^2-1\right)r\cos\theta}{\sqrt{2-\bar{a}^2}}+\bar{a}r\sin\theta\Big)\\
&\Big(-\bar{a}w-\bar{a}\sqrt{2-\bar{a}^2}r\cos\theta+\left(\bar{a}^2-2\right)r\sin\theta\Big)\\
&+\gamma\Big(w+\sqrt{2-\bar{a}^2}\left(\bar{a}^2-1\right)r\cos\theta-\bar{a}\left(\bar{a}^2-2\right)r\sin\theta\Big)\Big)+O(\varepsilon^2)\\
= \; & \varepsilon F_2(\theta,r,w)+O(\varepsilon^2).
\end{aligned} \tag{15}
$$

We shall apply the averaging theory described in Theorem 4 to the differential system (15). *Using the notation of section 3 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=(r,w)^T$ and*

$$
F(\theta,r,w)=\begin{pmatrix} F_1(\theta,r,w)\\ F_2(\theta,r,w)\end{pmatrix}, \quad \textit{and} \quad f(r,w)=\begin{pmatrix} f_1(r,w)\\ f_2(r,w)\end{pmatrix}.
$$

It is immediate to check that system (14) *satisfies all the assumptions of Theorem 4.*

Now we compute the integrals (9)*, i.e.*

$$
\begin{aligned}
f_1(r,w) = \; & \frac{1}{2\pi}\int_0^{2\pi}F_1(\theta,r,w)\mathrm{d}\theta\\
= \; & \frac{r\big(2(\alpha-\gamma)+\bar{a}(\bar{a}(-3\alpha+\bar{a}(-w+\beta+\bar{a}(\alpha-\gamma))+3\gamma)-2\beta)\big)}{2\left(2-\bar{a}^2\right)^{5/2}},\\
f_2(r,w) = \; & \frac{1}{2\pi}\int_0^{2\pi}F_2(\theta,r,w)\mathrm{d}\theta\\
= \; & \frac{2w(\gamma-\alpha)\bar{a}^2+2\left(r^2+w(w+2\beta)\right)\bar{a}+4w(\alpha-\gamma)-\left(r^2+2w\beta\right)\bar{a}^3}{2\left(2-\bar{a}^2\right)^{5/2}}.
\end{aligned}
$$

The system $f_1(r,w) = f_2(r,w) = 0$ *has a unique solution* (r^*, w^*) *with* $r^* > 0$, *namely*

$$\begin{aligned} r^* &= \frac{\sqrt{2(\bar{a}^2-2)(-\alpha+\bar{a}(1-\bar{a}^2)\beta+\gamma)((\bar{a}^2-1)\alpha+\bar{a}\beta+(1-\bar{a}^2)\gamma)}}{\bar{a}^3}, \\ w^* &= \frac{\left(\bar{a}^2-2\right)\left(\left(\bar{a}^2-1\right)\alpha+\bar{a}\beta+\left(1-\bar{a}^2\right)\gamma\right)}{\bar{a}^3}, \end{aligned}$$

if $\bar{a} \neq 0$ *and*

$$\left(-\alpha+\bar{a}\left(1-\bar{a}^2\right)\beta+\gamma\right)\left(\left(\bar{a}^2-1\right)\alpha+\bar{a}\beta+\left(1-\bar{a}^2\right)\gamma\right) < 0.$$

The Jacobian (10) *at* (r^*, w^*) *takes the value*

$$-\frac{(\alpha+\bar{a}\beta-\gamma)\left(-\alpha+\bar{a}\left(1-\bar{a}^2\right)\beta+\gamma\right)}{2\left(\bar{a}^2-2\right)^3}.$$

Moreover the eigenvalues of the Jacobian matrix

$$\left.\frac{\partial(f_1,f_2)}{\partial(r,w)}\right|_{(r,w)=(r^*,w^*)}$$

are the ones given in (5). *In short, the rest of the proof of the theorem follows immediately from Theorem 4 if we show that the periodic solution corresponding to* (r^*, w^*) *provides a periodic orbit bifurcating from the origin of coordinates of the differential system* (11) *at* $\varepsilon = 0$.

Theorem 4 garantes for $\varepsilon > 0$ *sufficiently small the existence of a periodic solution* $(r(\theta,\varepsilon), w(\theta,\varepsilon))$ *of system* (15) *such that* $(r(0,\varepsilon), w(0,\varepsilon)) \to (r^*, w^*)$ *when* $\varepsilon \to 0$. *So, system* (14) *has the periodic solution*

$$\big(u(\theta,\varepsilon) = r(\theta,\varepsilon)\cos\theta, v(\theta,\varepsilon) = r(\theta,\varepsilon)\sin\theta, w(\theta,\varepsilon)\big), \tag{16}$$

for $\varepsilon > 0$ *sufficiently small. Consequently, system* (12) *has the periodic solution* $(X(\theta), Y(\theta), Z(\theta))$ *obtained from* (16) *through the change of variables* (13). *Finally, for* $\varepsilon > 0$ *sufficiently small system* (11) *has a periodic solution* $(x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))$ *which tends to the origin of coordinates when* $\varepsilon \to 0$. *Therefore, it is a periodic solution starting at the zero-Hopf equilibrium point located at the origin of coordinates when* $\varepsilon = 0$. *This completes the proof of the theorem.*

Proof 7 (Proof of Theorem 3) . *If* $(a,b,c) = (\varepsilon\alpha, \bar{b}+\varepsilon\beta, \varepsilon\gamma$ *with* $\varepsilon > 0$ *a sufficiently small parameter, then the Rössler system becomes*

$$\begin{aligned} \dot{x} &= -y-z, \\ \dot{y} &= x+\varepsilon\alpha y, \\ \dot{z} &= (\bar{b}+\varepsilon\beta)x-\varepsilon\gamma z+xz. \end{aligned} \tag{17}$$

Doing the rescaling of the variables $(x,y,z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)$, *and denoting* *again the variables* (X,Y,Z) *by* (x,y,z) *system* (17) *writes*

$$\begin{aligned}\dot{x} &= -y-z,\\ \dot{y} &= x+\varepsilon\alpha y,\\ \dot{z} &= \bar{b}x+\varepsilon(\beta x-\gamma z+xz).\end{aligned} \tag{18}$$

We shall write the linear part at the origin of the differential system (18) *when* $\varepsilon=0$ *into its real Jordan normal form, i.e. as*

$$\begin{pmatrix} 0 & -\sqrt{\bar{b}+1} & 0\\ \sqrt{\bar{b}+1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

For doing that we consider the linear change $(x,y,z)\to(u,v,w)$ *of variables given by*

$$x=v,\quad y=-\frac{\sqrt{\bar{b}+1}\,u+w}{\bar{b}+1},\quad z=\frac{w-\bar{b}\sqrt{\bar{b}+1}\,u}{\bar{b}+1}.$$

In the new variables (u,v,w) *the differential system* (12) *writes*

$$\begin{aligned}\dot{u} =&\ -\sqrt{\bar{b}+1}v+\varepsilon\frac{\sqrt{\bar{b}+1}u\alpha-v(w+\beta)+\bar{b}\left(\sqrt{\bar{b}+1}u(v-\gamma)-v\beta\right)+w(\alpha+\gamma)}{(\bar{b}+1)^{3/2}},\\ \dot{v} =&\ \sqrt{\bar{b}+1}u,\\ \dot{w} =&\ \varepsilon\frac{v(w+\beta)-w\gamma+\bar{b}\left(w\alpha+v\beta+\sqrt{\bar{b}+1}u(-v+\alpha+\gamma)\right)}{\bar{b}+1}.\end{aligned} \tag{19}$$

Writing the differential system (19) *in cylindrical coordinates* (r,θ,w) *defined by* $u=r\cos\theta$ *and* $v=r\sin\theta$, *and we obtain*

$$\begin{aligned}\frac{dr}{d\theta} =&\ \varepsilon\frac{\cos\theta\left(w(\alpha+\gamma)-r(w+\bar{b}\beta+\beta)\sin\theta+\sqrt{\bar{b}+1}r\cos\theta(\alpha-\bar{b}\gamma+\bar{b}r\sin\theta)\right)}{(\bar{b}+1)^2}\\ &+O(\varepsilon^2)\\ =&\ \varepsilon F_1(\theta,r,w)+O(\varepsilon^2),\\ \frac{dw}{d\theta} =&\ \varepsilon\frac{w(\bar{b}\alpha-\gamma)+r(w+\bar{b}\beta+\beta)\sin\theta-\bar{b}\sqrt{\bar{b}+1}r\cos\theta(-\alpha-\gamma+r\sin(\theta))}{(\bar{b}+1)^{3/2}}\\ &+O(\varepsilon^2)\\ =&\ \varepsilon F_2(\theta,r,w)+O(\varepsilon^2).\end{aligned} \tag{20}$$

It is immediate to check that system (20) *satisfies all the assumptions of Theorem 4.*

Now we compute the integrals (9)*, i.e.*

$$f_1(r,w) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta,r,w)\mathrm{d}\theta = \frac{r(\alpha-\bar{b}\gamma)}{2(\bar{b}+1)^{3/2}},$$

$$f_2(r,w) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta,r,w)\mathrm{d}\theta = \frac{w(\bar{b}\alpha-\gamma)}{(\bar{b}+1)^{3/2}}.$$

The system $f_1(r,w) = f_2(r,w) = 0$ *has a unique solution* $(0,0)$ *and consequently the averaging theory does not provide any information about the possible periodic orbits which can bifurcate from the zero–Hopf equilibrium in this case.*

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