# THE GENERALIZED KITE PROBLEM


LUIS FRANCO-PÉREZ[1], ERNESTO PÉREZ-CHAVELA[2], SŁAWOMIR RYBICKI[3]

[1]*Departamento de Matemáticas Aplicadas y Sistemas ,*
*UAM-Cuajimalpa, Av. Vasco de Quiroga 4871, México, DF 05348 México*
*Email: lfranco@correo.cua.uam.mx*
[2]*Departamento de Matemáticas,*
*UAM-Iztapalapa, Av. Rafael Atlixco 186, México, DF 09340, México*
*Email: epc@xanum.uam.mx*
[3]*Faculty of Mathematics and Computer Science*
*Nicolaus Copernicus University , PL-87-100 Toruń, ul. Chopina* 12/18 *, Poland*
*Email: rybicki@mat.umk.pl*



*Abstract.* We study a particular planar four body problem with three degrees of freedom, where the particles move under the influence of a potential $U_\alpha$ such that for $\alpha = 1$ we recover the classical Newtonian Potential. We describe the topology of the manifolds of constant energy, for any value of the total energy $h$. We also describe the total collision manifold and its topology.



*Key words*: four body problem, kite.


*To Vasile Mioc, an excellent astronomer,*
*mathematician and person.*

## 1. INTRODUCTION

Celestial mechanics is a branch of mathematics which has motivated the development of many others areas of mathematics, and continue with this tendency attracting many people from other fields like physics, astronomy and astrophysics principally. In this work we have tackled a planar four body problem where we impose some constraints in the initial conditions (positions and velocities), in order to maintain always a kite shape configuration, which could be concave, if one mass is located in the interior of the convex hull of the other three masses, or convex if this does not happen. Additionally we work with an attractive potential given for a homogeneous function $U_\alpha$ of degree $-\alpha$ with $\alpha > 0$, in such a way that for $\alpha = 1$ we recover the classical Newtonian Potential. The problem has three degrees of freedom and it posses singularities due to binary collisions, simultaneous binary collisions, two kind of triple collisions and the total collision singularity. The above shows in part the complexity of this problem, understand completely it is a really difficult task, the problem is at least as complicated as the general three body problem.

The idea to study the global dynamics of this problem comes from the works of Albouy (1995); Moeckel (1985); Pérez-Chavela and Santoprete (2007), where they study central configurations in problems with several symmetries. A central configuration is a particular position of the masses where the position and acceleration vectors are proportional, with the same constant of proportionality for all masses. They generate the relative equilibria solutions, i.e. solutions where the system behaves as a rigid body problem, and in general they generate the homographic solutions, the unique explicitly solutions of the $N$–body problem know until now, where the particles have a similar configuration (a central configuration) for all time. This work represents the first attempt to study some aspects of the global dynamics in this problem.

The goal of this paper is essentially motivational, as in many papers of Vasile Mioc (*e.g.* Mioc and, Pérez-Chavela (2005)), we try to motivate students and young researchers in this problem. Here we give the first steps to understand it, in Section 2 we give the equations of motion and explain in detail the symmetries and restrictions of the problem. We think that this is a very important first step, give the model with the respective equations of motion. In Section 3 we give the topology of the energy levels in the general problem, without any regularization of the singularities. In Section 4, we introduce McGehee-type coordinates to blow up the singularity due to total collision McGehee (1974), in other words, instead to regularize the total collision we glue a submanifold which appears as the border of all energy levels, in this way we introduce new coordinates and a new reparametrization of time. The total collision occurs now in infinity time, so the analysis of asymptotic motions tending to the total collision singularity, which is a submanifold of codimension 1 in each energy level, give us important information about the motions close to total collision. In Section 5, we introduce the concept of central configurations, and we mention some important properties of them, in particular the homothetic orbits for the global flow, a particular ejection-collision orbit getting from the heteroclinic connection between two equilibrium points. We also discuss about the fact that the global flow is projectable on the total collision manifold. We finish this paper giving a sequence of open questions that we hope could be of interest for some readers.

## 2. EQUATIONS OF MOTION

We consider four point positive masses on a plane possessing a symmetry line containing two of the particles with masses $m_1$ and $m_2$, we suppose that the mass $m_1$ is always above the mass $m_2$; the other two masses that must be equal in order to preserve the symmetry are located at the same distance of the symmetry line, that is on a line which is perpendicular to it. Let $\mu$ be the common value of these masses. We give initial conditions in positions and velocities in such a way that the particles preserve the symmetry line, we call this problem *the kite problem*. The kite configuration is *convex* if none of the bodies is located in the interior of the convex hull of the other three, otherwise and if the configuration is not collinear we say that the kite configuration is *concave*.

Let $x$ be the semi-distance between the particles of mass $\mu$, $y$ the distance between the particles with masses $m_1$ and $m_2$ and $z$ the distance from the particle with mass $m_1$ to the intersection of the symmetry line with the line containing the particles of mass $\mu$. We take this last distance with sign, it is positive if $m_1$ is above the line containing the particles with masses $\mu$, and negative if it is below (see Figure 1). The particular case $y = 2z$ entails that $m_1 = m_2$, this problem is known as the rhomboidal four body problem, it has been widely studied in Lacomba and Pérez-Chavela (1992) and Lacomba and Pérez-Chavela (1993). The configuration space $CS$ is the open set given by $CS = \{q = (x,y,z) | x > 0, y > 0, z \in \mathbb{R}\}$.

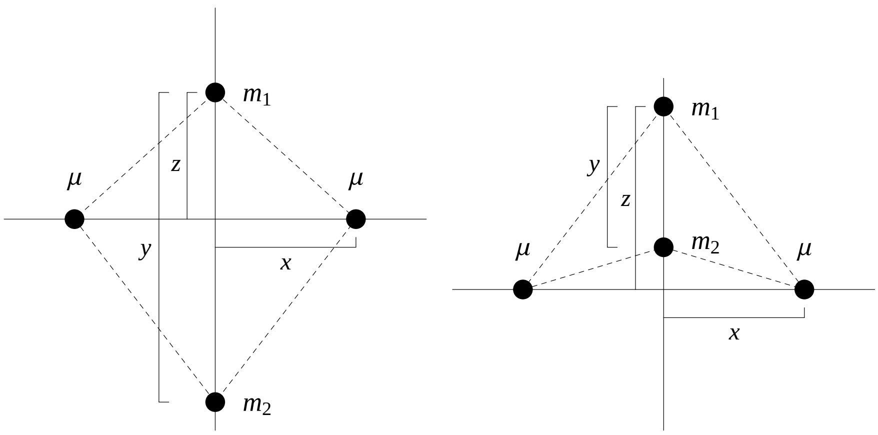



Fig. 1 – Convex and concave configuration.

In the above coordinates the Lagrangian of the system can be written as

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{m_2(2\mu+m_1)\dot{y}^2 - 4\mu m_2 \dot{y}\dot{z} + 2\mu(m_1+m_2)\dot{z}^2}{2\Sigma} + \mu \dot{x}^2 + U_\alpha(x,y,z), \quad (1)$$

where

$$U_\alpha(x,y,z) = \frac{m_1 m_2}{y^\alpha} + \frac{\mu^2}{(2x)^\alpha} + \frac{2\mu m_1}{(x^2+z^2)^{\alpha/2}} + \frac{2\mu m_2}{(x^2+(y-z)^2)^{\alpha/2}}, \quad (2)$$

and $\Sigma = m_1 + m_2 + 2\mu$. We observe that for the special case $\alpha = 1$ we recover the classical Newtonian potential. In general we will be working with a potential which is a homogeneous function of degree - $\alpha$, and since we are interested just in attractive forces we assume from here on that $\alpha > 0$. We call this problem *the generalized kite problem*, since for $\alpha = 1$ it is known as the kite problem.

It is not difficult to verify that the mass matrix has the form

$$M = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & \frac{m_2(2\mu+m_1)}{\Sigma} & \frac{-2\mu m_2}{\Sigma} \\ 0 & \frac{-2\mu m_2}{\Sigma} & \frac{2\mu(m_1+m_2)}{\Sigma} \end{pmatrix},$$

which by short we write as

$$M = \begin{pmatrix} 2\mu & 0 \\ 0 & C \end{pmatrix},$$

after some computations we obtain

$$M^{-1} = \begin{pmatrix} \frac{1}{2\mu} & 0 \\ 0 & C^{-1} \end{pmatrix},$$

where

$$C^{-1} = \begin{pmatrix} \frac{m_1+m_2}{m_1 m_2} & \frac{1}{m_1} \\ \frac{1}{m_1} & \frac{2\mu+m_1}{2\mu m_1} \end{pmatrix}.$$

With the above notations we can write the Hamiltonian of the system in a compact form as:

$$E(q,p) = \frac{1}{2}pM^{-1}p^t - U_\alpha(q), \tag{3}$$

where $p = M\dot{q}$. The equations of motion are given by

$$\begin{cases} \dot{q} = M^{-1}p^t = \partial E/\partial p, \\ \dot{p} = \nabla U_\alpha(q) = -\partial E/\partial q, \end{cases} \tag{4}$$

the energy relation takes the form

$$E = h \Longleftrightarrow \frac{1}{2}pM^{-1}p^t = U_\alpha(q) + h. \tag{5}$$

In this problem the total angular momentum is zero. The configuration space $CS$ is formed by the two octants mentioned before. The set of singularities form the frontier of $CS$. The half planes defined by $\{(x,y,z)|x = 0, y > 0, z \in \mathbb{R}\}$ and $\{(x,y,z)|x > 0, y = 0, z \in \mathbb{R}\}$ arise as boundary components of $CS$, representing binary collisions between the two particles with mass $\mu$ and the ones with masses $m_1$ and $m_2$. The two semi rays given by $x = 0, y = 0 \,\&\, z \neq 0$ represent the set of simultaneous binary collisions, a concept which in general is a big challenge to understand. The semi ray $x = 0, y \neq 0 \,\&\, z = 0$ represents the triple collision among the particles of mass $\mu$ with $m_1$. The semiray $x=0,\ y=z\neq 0$ represents the triple collision among the particles of mass $\mu$ with $m_2$. Finally the origin $(x,y,z)=(0,0,0)$ represents the quadruple or total collision (see Figure 2). The total understanding of the above singularities is a difficult challenge. In this short paper we are interested in the study of the topologies of the energy levels and on the regularization of the singularity due to total collision, or more precisely in the blow-up of the total collision.

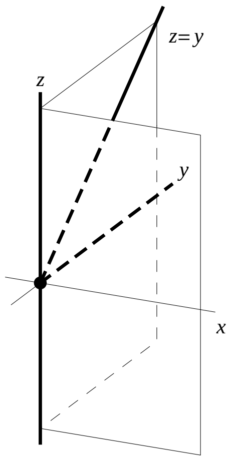



Fig. 2 – Frontier of $CS$. Half planes stand for binary collisions, thick dark lines stand for triple and simultaneous binary collisions and the big dot stands for total collision.

### 3. TOPOLOGY OF THE ENERGY LEVELS

We know that if $h$ is a regular value of the total energy function, then the corresponding energy level denoted by $E_h=\{(q,p)\,|\,\frac{1}{2}pM^{-1}p^t=U_\alpha(q)+h\}$ is a manifold of dimension 5 embedding in $\mathbb{R}^6$. It is important to remark the dimension of the respective spaces since even that we can obtain the topology of the energy levels, the shape of them could be really complicated.

**Theorem 1.** *With the above notations, assuming that $h$ is a regular value, each energy level is topologically equivalent to*

$$\begin{cases} E_h\approx\mathbb{R}^2\times D^3 & \text{if} \quad h<0,\\ E_h\approx\mathbb{R}^3\times S^2 & \text{if} \quad h\geq 0.\end{cases} \tag{6}$$

*Proof.* We divide the proof in two cases, depending of the sign of the energy

– If $h<0$, then for each $q$ such that $U_\alpha(q)>-h$ we have that $\frac{1}{2}pM^{-1}p^t=$

$U_\alpha(q)+h$ is topologically a sphere $S^2$. When $U_\alpha(q)=-h$, it becomes a point, so $E_h$ is a pinched $S^2$ bundle over $\{q|U_\alpha(q)\geq -h\}$, which by short we call it as $D^3$. Observe that this is a 3-dimensional set. Now since $\partial U_\alpha/\partial x<0\,\forall\, x\in\mathbb{R}^+$, we can solve $x$ in terms of $y$ and $z$, therefore the configuration space is homeomorphic to an open set of $\mathbb{R}^2$, which is homeomorphic to $\mathbb{R}^2$. With all the above we obtain $E_h\approx\mathbb{R}^2\times D^3$.

– If $h\geq 0$, then we do not have any restriction on the configuration space, that is for each $q$ we have that $\frac{1}{2}pM^{-1}p^t=U_\alpha(q)+h$ is topologically a sphere $S^2$. Therefore $E_h\approx\mathbb{R}^3\times S^2$.

□

## 4. MCGEHEE COORDINATES

Let $\rho=\sqrt{qMq^t},\quad Q=\rho^{-1}q,\quad P=(\rho)^{\alpha/2}p,\quad dt=\rho^{\alpha/2+1}d\tau$. The equations of motion (4) take the form

$$\begin{aligned}\rho^{'} &= \lambda\rho,\\ Q^{'} &= M^{-1}P-\rho^\alpha\lambda Q,\\ P^{'} &= \nabla U_\alpha(Q)+\frac{\alpha}{2}\lambda P.\end{aligned}\tag{7}$$

Where $'$ means derivation with respect to the new time $\tau$ and $\lambda=P\cdot Q$. We note that $QMQ^t=1$. The energy relation (5) takes the form

$$\frac{1}{2}PM^{-1}P^t=U_\alpha(Q)+\rho h.\tag{8}$$

Since $\rho^{'}=0$ when $\rho=0$ we obtain an invariant set by the flow given by (7) called the total collision manifold, that is given by

$$\Lambda=\left\{(\rho,Q,P)\,\middle|\,\frac{1}{2}PM^{-1}P^t=U_\alpha(Q),\,Q\in B\right\},\tag{9}$$

where $B=\{Q\in CS\,|\,QMQ^t=1\}$. It forms a submanifold of $E_h$ of dimension 4.

**Theorem 2.** *The total collision manifold $\Lambda$ is topologically equivalent to $S^2\times\mathbb{R}^2$.*

*Proof.* For $\alpha>0$, the potential $U_\alpha$ is positive over $B$, thus $PM^{-1}P^t=c$ is a 2-sphere when $U_\alpha(Q)=c$. Since $B\subset CS$, it is an open branch of the ellipsoid $QMQ^t=1$, that is

$$B=\{Q=(Q_1,Q_2,Q_3)\,|\,Q_1=h(Q_2,Q_3),Q_2\in(0,a),Q_3\in(-b,b)\}$$

for some $a, b > 0$ and

$$h(Q_2, Q_3) = \frac{1}{\sqrt{2\mu}} \sqrt{1 - \frac{4\mu m_2 \delta^2 \sigma^2 Q_2 Q_3 - \sigma^2 \Sigma - \delta^2 \Sigma}{\sigma^2 \delta^2 \Sigma}},$$

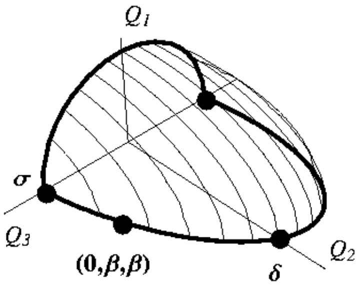
where

$$\delta = \sqrt{\frac{\Sigma}{(2\mu + m_1)m_2}}, \qquad \sigma = \sqrt{\frac{\Sigma}{2\mu(m_1 + m_2)}}.$$

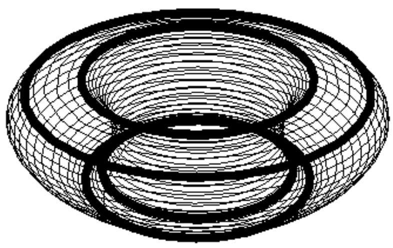
Therefore, $B$ is homeomorphic to the open square $(0,a) \times (-b,b)$, which is equivalent to $\mathbb{R}^2$.

So $\Lambda \approx S^2 \times \mathbb{R}^2$. □

Total collision has been blown up, set $\Lambda$, and knowing the flow close to it, we obtain much information about dynamics close to total collision, since continuity of the flow with respect to initial data. Since $\Lambda$ is invariant by the total flow, we restrict for a while the analysis of the dynamics on it. The configuration space on $\Lambda$ is the open set $B$, a 2-dimensional submanifold. The boundary of $B$ are formed by the set of singularities representing the binary and triple collisions (see Figure 3). Binary collisions between the masses $\mu$ are given by the curve $Q_1 = 0$ and the ones between the masses $m_1$ and $m_2$ are defined by the curve $Q_2 = 0$. Simultaneous binary collisions are indicated by the points $(0,0,\sigma)$ and $(0,0,-\sigma)$. Triple collisions are determined by the point $(0,\delta,0)$ for the masses $\mu$ with the mass $m_1$ and by the point $(0,\beta,\beta)$ with $\beta = (\Sigma/(2\mu m_1 + m_1 m_2))^{1/2}$ for the masses $\mu$ with the mass $m_2$. A projection on $S^1 \times \mathbb{R}^2$ of the total collision manifold $\Lambda$ is showed in Figure 3. The boundary represents binary collisions and dark curves stand for simultaneous binary and triple collisions.




(a)



(b)

Fig. 3 – Components for the total collision manifold. (a) Set $B$: dark lines stand for binary collisions and big dots for simultaneous binary and triple collisions. (b) Projection of $\Lambda$ on $S^1 \times \mathbb{R}^2$.

## 5. CENTRAL CONFIGURATIONS

The equilibrium points of equations (7) are always on the total collison manifold $\Lambda$ since all of them must satisfy that $\rho = 0$. We also observe that the momenta coordinate $P = 0$ and then $\nabla U_\alpha(Q) = 0$, in other words the equilibrion points are associated to the central configurations. A central configuration is a particular position of the masses where the position and acceleration vectors are proportional, with the same constant of proportionality. They generate the unique explicitly solutions of the $N$–body problem known until now. It is well known that the central configurations correspond to the critical points of the potential restricted to a fix size (constant moment of inertia), that is to the algebraic solutions of $\nabla U_\alpha(Q) = 0$ (see Wintner (1941) for more details). For each central configuration we get two equilibrium points on $\Lambda$, and one heteroclinic orbit joint them, the so called homothetic orbit which is an ejection-collision orbit (see Devaney (1981) for more details).

From equation (8), we observe that for $h = 0$ we obtain the same equation which defines the total collision manifold $\Lambda$. This means that if one knows the global flow on $\Lambda$, then we also know the total global flow on this energy level, because we obtain the coordinate $\rho(t)$ integrating the first equation in (8), in this case we say that the flow is projectable, in other words all orbits of the global flow project through the coordinate $\rho$ to the respective trajectory on $\Lambda$.

For $\alpha = 1$, there is always one non-degenerate convex central configuration for the kite problem Pérez-Chavela and Santoprete (2007). If the four masses are equal the above central configuration is a square and additionally, if three of the masses are at the vertices of an equilateral triangle and the fourth particle with arbitrary mass is at the baricenter, they form a central configuration. We also know that there is another concave central configuration forming an isosceles triangle shape with the forth mass on the axis of symmetry (of course we have to count also the reflected ones). For $\alpha = 1$ we know all central configurations Pérez-Chavela and Santoprete (2007), an interesting question out of the goal of this paper is to find all central configurations for any value of the parameter $\alpha$.

## 6. FINAL REMARKS AND OPEN QUESTIONS

In section 4 we have regularized the singularity due to total collision, in fact, properly speaking, we have blow-up the singularity due to total collision, generating the total collision manifold $\Lambda$. We have seen in the previous section that the equilibrium points of equations (7) are associated to the central configurations. For $\alpha = 1$, the classical kite problem, it is known that there exists a unique convex central configuration for all choices of the masses, from here we have our first open question, by short OQ.

OQ1 Prove that for any value of positive $\alpha$ there exists a unique convex central configuration for any choice of the masses. And in general find all central configurations for any values of the parameter $\alpha$.

The next OQ are related with the regularization of the different kinds of singularities.

OQ2 Regularize the simple binary collisions and determine the topology of the energy surfaces and total collision manifold taken into account these regularization. Does the regularization change for for different values of $\alpha$? Of course the regularization depends on $\alpha$?, the question is if the rule is the same, or if it changes for some values of $\alpha$ and if there are some kind of bifurcations.

OQ3 Regularize the simultaneous binary collisions. Even in the case of $\alpha = 1$, this is an interesting question. How the regularization of the simultaneous binary collisions depends on the value of the parameter $\alpha$ (The readers interested must consult the work of Martínez-Simó Martínez and Simó (1999)).

The next two questions are more complicated, there are not any references to tacked them, actually the last one depends of the answer to the previous questions.

OQ4 Regularize the two kinds of triple collision singularities, how the regularization depends on $\alpha$. It is possible regularize both kinds of triple collision in just one step as in the case of the simple binary collisions?

OQ5 Give a total description of the topology of the regularized energy levels.

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