

LAGRANGIAN AND HAMILTONIAN MECHANICS WITH FRACTIONAL DERIVATIVES

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Abstract. In this paper we discuss the fractional extension of classical Lagrangian and Hamiltonian mechanics. We give a view of the mathematical tools associated with fractional calculus as well as a description of some applications.

Key words: fractional derivative - Lagrangian - Hamiltonian - Euler-Lagrange equations.

1. INTRODUCTION

Lagrangian mechanics and Hamiltonian mechanics are alternative formulations of classical Newtonian mechanics. Their importance is represented by the fact that any of them could be used to solve a problem in classical mechanics. We emphasize that the Newtonian mechanics requires the concept of force, while Lagrangian and Hamiltonian systems are expressed in terms of energy.

For a system of n interacting particles $m_i > 0$, $i = 1, 2, \dots, n$, let $\mathbf{q}_i = (x_i, y_i, z_i)$, $\mathbf{q}_i \in \mathbf{R}^3$ be their position vectors with respect to an arbitrary origin and let

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \in \mathbf{R}^{3n}$$

be the configuration of the system, $\mathbf{q}_i = \mathbf{q}_i(t)$. We have verified the Newton's second law

$$\mathbf{F}_i = m_i \ddot{\mathbf{q}}_i,$$

for each particle i , $i = 1, 2, \dots, n$, where \mathbf{F}_i is the force on the particle i . The linear momentum of the system is

$$\mathbf{p} = \sum_{i=1}^n m_i \dot{\mathbf{q}}_i$$

and the kinetic energy is

$$K = \frac{1}{2} \sum_{i=1}^n m_i \|\dot{\mathbf{q}}_i\|^2,$$

where $\|\dot{\mathbf{q}}_i\|^2 = \dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i$ is the Euclidian norm.

The dynamics of this n -body system in the field deriving from the function potential $V(\mathbf{q})$ is

$$m_i \ddot{\mathbf{q}}_i = -\frac{\partial V}{\partial \mathbf{q}_i}, \quad i = 1, 2, \dots, n.$$

In this case, the total energy $E := K + V$ is conserved and the system is called conservative. This system of equations is equivalent to the Euler-Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial L}{\partial \mathbf{q}_i} = 0, \quad i = 1, 2, \dots, n$$

for the Lagrangian

$$L(t, \mathbf{q}, \dot{\mathbf{q}}) := \frac{1}{2} \sum_{i=1}^n m_i \|\dot{\mathbf{q}}_i\|^2 - V(\mathbf{q}).$$

We remark that $L = K - V$.

The following variational principle, named Hamilton's principle of least action, is valid: for any differentiable Lagrangian L , the Euler-Lagrange equations are equivalent to $\delta S = 0$, where

$$S[\mathbf{q}] := \int_a^b L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) dt$$

and δS is the first variation of S ,

$$\delta S = \frac{\partial}{\partial s} \int_a^b L(t, \mathbf{q}(t, s), \dot{\mathbf{q}}(t, s)) dt \Big|_{s=0},$$

for a deformation $\mathbf{q}(t, s)$ of $\mathbf{q}(t)$ leaving the endpoints fixed.

The above system of equations is equivalent to Hamilton's equations,

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

for the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) := \frac{1}{2} \sum_{i=1}^n \frac{1}{m_i} \|\mathbf{p}_i\|^2 + V(\mathbf{q}),$$

where $\mathbf{p}_i = m_i \dot{\mathbf{q}}_i$, $i = 1, 2, \dots, n$ and $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$. For such a \mathbf{p} , we have $H(\mathbf{q}, \mathbf{p}) = K + V$.

We can also define Hamilton's equations on \mathbf{R}^{3n} for any H , not necessarily

derived from a Lagrangian:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}},$$

for some $H : \mathbf{R}^{3n} \times \mathbf{R}^{3n} \rightarrow \mathbf{R}$, called Hamiltonian. Of course, the Hamiltonian is conserved.

The Euler-Lagrange equations and Hamilton's principle form the basis of Lagrangian (or Hamiltonian) mechanics. The power of Lagrangian (Hamiltonian) mechanics is that the given equations are characterized with only one scalar functional, the Lagrangian L , or the Hamiltonian H . Generally, these functionals only describe conservative systems. There have been some approaches at describing non-conservative systems in such formalism. The method presented by Rayleigh introduce a functional R (called Rayleigh's dissipation function) and rewrite the Euler-Lagrange equations in the following form:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}_i} \right) - \frac{\partial L}{\partial \mathbf{q}_i} + \frac{\partial R}{\partial \dot{\mathbf{q}}_i} = 0, \quad i = 1, 2, \dots, n,$$

where L is the system's Lagrangian. These equations provide a way to treat dissipative systems, but by means of two scalar functionals, not one. This is an objection to this method.

Riewe proposed an approach to incorporate non-conservative systems to the field of variational mechanics using fractional calculus (Riewe, 1996; Riewe, 1997). He has shown that Lagrangian with fractional derivative lead to equations of motion with non-conservative classical forces such as friction and this formalism could be applied to a fractional force proportional to the velocity. On the other hand, Riewe (1997) presented a method to obtain potentials for non-conservative forces in order to introduce dissipative effects to the Lagrangian and Hamiltonian mechanics.

The Hamiltonian and Lagrangian involving fractional derivative is also used to derive the equation damped harmonic oscillator (see Tarawneh *et al.*, 2010). Therefore, the dynamical systems with fractional order can be dissipative. For this reason, the theory and methods of fractional calculus are extensively used for describing critical phenomena in non-equilibrium systems of physics and mechanics, especially in the complex systems.

2. FRACTIONAL DERIVATIVES

Fractional calculus, which generalize the classical calculus, is the theory of derivatives and integrals of arbitrary non-integer order. In the last years interest in fractional calculus has been stimulated by the applications in different areas of science and engineering. In the mathematical modelling of many systems and processes,

the new fractional-order models are more adequate than the integer-order models. In the recent years, fractional derivatives have played a important role in very diverse topics such as classical mechanics, scaling phenomena, fractals and multi-fractals dynamics, dispersion and turbulence, astrophysics, potential theory, viscoelasticity, electrodynamic, optics, and thermodynamic.

In the sequel, we give a description of the basic concepts of fractional calculus. Several definitions of a fractional derivative have been proposed, as Riemann–Liouville, Grunwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivatives (see Samko *et al.*, 1993; Podlubny, 1999; Hilfer, 2000; Kilbas *et al.*, 2006; Hilfer, 2008). We present two definitions: Riemann-Liouville fractional derivative and Caputo fractional derivative.

Let $f : [a, b] \rightarrow \mathbf{R}$ be a function, where a and b can even be infinite. The left Riemann-Liouville fractional derivative (with fixed lower terminal a and moving upper terminal t) is defined by

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(u)}{(t-u)^{\alpha+1-n}} du$$

and the right Riemann-Liouville fractional derivative (with moving lower terminal t and fixed upper terminal b) is defined by

$${}_t D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_t^b \frac{f(u)}{(u-t)^{\alpha+1-n}} du,$$

where $n-1 \leq \alpha < n$, Γ represents the Euler gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

and $\left(\frac{d}{dt} \right)^n$ stands for ordinary derivatives of integer order n . Alternative definitions of Riemann-Liouville fractional derivatives are Caputo derivatives. The left Caputo fractional derivative defined as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \left(\frac{d}{du} \right)^n f(u) du$$

and the right Caputo fractional derivative

$${}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (u-t)^{n-\alpha-1} \left(-\frac{d}{du} \right)^n f(u) du,$$

where the order α satisfies $n - 1 \leq \alpha < n$. The Riemann-Liouville derivative of constant is not zero, although Caputo derivative of a constant is zero. Following Podlubny (1999), the left derivative and the right derivative are operations performed on the past states, respectively on the future states, of the process f . If $u < t$, where t is the present moment, then the state $f(u)$ belongs to the past of the process f ; if $u > t$, then $f(u)$ belongs to the future of the process f . Thus, the present state of the process $f(t)$, which started at $u = a$, depends on all its past states $f(u)$, $a \leq u < t$.

If α is an integer, the Riemann-Liouville derivatives are defined in the usual sense, i.e.

$${}_a D_t^\alpha f(t) = \left(\frac{d}{dt} \right)^\alpha f(t), \quad {}_t D_b^\alpha f(t) = \left(-\frac{d}{dt} \right)^\alpha f(t),$$

where $\alpha = 1, 2, 3, \dots$

For $\alpha \in \mathbf{R}_+$, we can also define the operators ${}_a J_t^\alpha$ and ${}_t J_b^\alpha$ on $L^1([a, b])$:

$${}_a J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(u)}{(t-u)^{1-\alpha}} du$$

and

$${}_t J_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(u)}{(u-t)^{1-\alpha}} du.$$

These operators are called the left and the right fractional Riemann-Liouville integrals of order $\alpha \in \mathbf{R}_+$, respectively. It is easy to see that the Riemann-Liouville fractional integrals converge for any integrable function f . For integer α , $\alpha = n$, the fractional Riemann-Liouville integrals coincide with the usual integer order n -fold integration (Cauchy formula for n -fold integration):

$$\int_a^t \int_a^{t_n} \int_a^{t_{n-1}} \dots \int_a^{t_3} \int_a^{t_2} f(t_1) dt_1 dt_2 \dots dt_{n-1} dt_n = \frac{1}{\Gamma(n)} \int_a^t \frac{f(u)}{(t-u)^{1-n}} du.$$

The integration operators ${}_a J_t^\alpha$ and ${}_t J_b^\alpha$ play a role in the definition of fractional calculus. The left and the right Riemann-Liouville fractional derivative of order $\alpha > 0$ are

$${}_a D_t^\alpha f(t) = D_t^n {}_a J_t^{n-\alpha} f(t), \quad {}_t D_b^\alpha f(t) = (-1)^n D_t^n {}_t J_b^{n-\alpha} f(t),$$

with $n = [\alpha] + 1$ and D_t^n is the ordinary derivative of integer order n . The left and the right Caputo fractional derivatives of order $\alpha \in \mathbf{R}_+$ are

$${}^C D_t^\alpha f(t) = {}_a J_t^{n-\alpha} D_t^n f(t), \quad {}^C D_b^\alpha f(t) = (-1)^n {}_t J_b^{n-\alpha} D_t^n f(t).$$

We observe that the formula of fractional derivative involves an integration which is a non-local operator, so fractional derivative is a non-local operator. Thus, calculating time-fractional derivative of a function $f(t)$ at some time t , it is required all the previous history from 0 to t . Time-fractional derivatives are naturally related to systems with memory. These systems are closely related to fractals, which are present in most physical systems.

A property of the fractional operator is

$${}_a D_t^p \left({}_a D_t^{-q} f(t) \right) = {}_a D_t^{p-q} f(t),$$

where f is continuous and $0 \leq q \leq p$. For $p > 0$ we obtain the fundamental property of Riemann-Liouville fractional derivative

$${}_a D_t^p \left({}_a D_t^{-p} f(t) \right) = f(t).$$

This formula means that the Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of some order. We also remark the following properties:

$${}_a D_t^\alpha f(t) = \frac{d^n}{dt^n} {}_a D_t^{\alpha-n} f(t), \quad {}_t D_b^\alpha f(t) = (-1)^n \frac{d^n}{dt^n} {}_t D_b^{\alpha-n} f(t),$$

$${}^C_a D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(a),$$

$${}^C_t D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \frac{(1-t)^{-\alpha}}{\Gamma(1-\alpha)} f(b).$$

The Mittag-Leffler functions E_α and $E_{\alpha,\beta}$ naturally occur in solutions of fractional order differential equations. Mittag-Leffler (1903) defined the function E_α as the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0$$

and Wiman (1905) obtained the generalisation of E_α , denoted $E_{\alpha,\beta}$ ($E_\alpha = E_{\alpha,1}$; $E_1 = \exp z$):

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \beta > 0.$$

3. FRACTIONAL EULER-LAGRANGE EQUATIONS

Riewe has used the fractional calculus to develop a formalism which can be used for both conservative and non conservative systems (see Riewe, 1996; Riewe, 1997). Using the fractional approach, one can obtain the Euler-Lagrange and the Hamiltonian equations of motion for the non-conservative systems. The classical calculus of variations was extended by Agrawal (2002) for systems containing Riemann-Liouville fractional derivatives. Mathematical tools analogous to calculus of variations will be needed to minimize certain functionals. Many of the concepts and results of classical calculus of variations can be extended with minor modifications to fractional calculus of variations. The fractional Euler-Lagrange equations are a set of differential equations involving both the left and the right fractional derivatives.

A fractional calculus of variations problem contains at least one fractional derivative term. We denote by \mathcal{F}_1 the set of all functions $q(t)$ which have continuous left Riemann-Liouville fractional derivative of order α and right Riemann-Liouville fractional derivative of order β for $x \in [a, b]$ and satisfy the conditions $q(a) = q_a$, $q(b) = q_b$. The problem can be defined as follows: find the function $q \in \mathcal{F}_1$ for which the functional

$$S[q] = \int_a^b L(t, q, {}_a D_t^\alpha q(t), {}_t D_b^\beta q(t)) dt$$

has an extremum, where $L(t, q, u, v)$ be a function with continuous first and second partial derivatives with respect to all its arguments.

A necessary condition for $S[q]$ to have an extremum for a given function $q(t)$ is Euler-Lagrange equation (Agrawal, 2002):

$$\frac{\partial L}{\partial q} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha q} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta q} = 0.$$

The generalized momenta are introduced as

$$p_\alpha = \frac{\partial L}{\partial {}_a D_t^\alpha q}, \quad p_\beta = \frac{\partial L}{\partial {}_t D_b^\beta q}$$

and the Hamiltonian depending on the fractional time derivatives is

$$H = p_\alpha {}_a D_t^\alpha q + p_\beta {}_t D_b^\beta q - L(t, q, {}_a D_t^\alpha q(t), {}_t D_b^\beta q(t)).$$

The Hamilton's equations of motion are obtained in a similar manner to the usual mechanics (Rabei *et al.*, 2007):

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \quad \frac{\partial H}{\partial p_\alpha} = {}_a D_t^\alpha q,$$

$$\frac{\partial H}{\partial p_\beta} = {}_t D_b^\beta q, \quad \frac{\partial H}{\partial q} = {}_t D_b^\alpha p_\alpha + {}_a D_t^\beta q p_\beta.$$

We remark that the fractional Hamiltonian is not a constant of motion even if the Lagrangian does not explicitly depend on the time.

When $\alpha = \beta = 1$,

$${}_a D_t^\alpha q = \frac{dq}{dt}, \quad {}_t D_b^\beta q = -\frac{dq}{dt}$$

the above functional reduces to the simplest form

$$S[q] = \int_a^b L(t, q, \dot{q}) dt$$

and Euler–Lagrange equation is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0.$$

We can generalize in a straight forward manner to problems containing several unknown functions. We denote by \mathcal{F}_n the set of all functions $q_1(t), q_2(t), \dots, q_n(t)$ which have continuous left Riemann-Liouville fractional derivative of order α and right Riemann-Liouville fractional derivative of order β for $x \in [a, b]$ and satisfy the conditions

$$q_i(a) = q_a^i, \quad q_i(b) = q_b^i, \quad i = 1, 2, \dots, n.$$

The problem can be defined as follows: find the functions q_1, q_2, \dots, q_n from \mathcal{F}_n , for which the functional

$$S[q_1, q_2, \dots, q_n] = \int_a^b L \left(t, q_1(t), q_2(t), \dots, q_n(t), {}_a D_t^\alpha q_1(t), \dots, {}_a D_t^\alpha q_n(t), {}_t D_b^\beta q_1(t), \dots, {}_t D_b^\beta q_n(t) \right) dt$$

has an extremum, where $L(t, q_1, \dots, q_n, u_1, \dots, u_n, v_1, \dots, v_n)$ is a function with continuous first and second partial derivatives with respect to all its arguments. A necessary condition for $S[q_1, q_2, \dots, q_n]$ to admit an extremum is that $q_1(t), q_2(t), \dots, q_n(t)$ satisfy Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_i} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha q_i} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta q_i} = 0 \quad i = 1, 2, \dots, n.$$

In vector notation, the above condition can be written as

$$\frac{\partial L}{\partial \mathbf{q}} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha \mathbf{q}} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta \mathbf{q}} = 0,$$

where $\mathbf{q} \in \mathbf{R}^n$.

4. SOME APPLICATIONS ON CONSERVATIVE AND NON-CONSERVATIVE SYSTEMS

A simple harmonic oscillator is a conservative system. This system consists of a force F which pulls the mass m in the direction of the point $x = 0$ and depends only on the mass's position x and a constant k . The Newton's second law for this system is

$$F = m \frac{d^2 x}{dt^2} = -kx.$$

The motion is described by the function $x(t) = A \cos(\omega t + \varphi)$, where

$$\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}.$$

The force F is conservative with the potential energy function

$$V(x) = \frac{1}{2} kx^2.$$

The Lagrangian of the particle can be written

$$L(t, x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x).$$

and the equations of motion are retrieved by applying the Euler–Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.$$

We have

$$\frac{\partial L}{\partial x} = -\frac{dV}{dx}, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

Thus

$$m\ddot{x} + \frac{dV}{dx} = 0, \quad m\ddot{x} + kx = 0.$$

We observe that if the Lagrangian of a system is known, then the equations of motion may be obtained by the Euler–Lagrange equations. The Lagrangian of a system is not unique. Lagrangians which describe the same system can differ by the total derivative with respect to time of some function, but they will give the same equations of motion.

We consider now a fractional Lagrangian of the above oscillatory system

$$L(t, x, {}_a D_t^\alpha x) = \frac{1}{2} m ({}_a D_t^\alpha x)^2 - \frac{1}{2} k x^2.$$

Then the fractional Euler–Lagrange equation is

$$m {}_t D_b^\alpha ({}_a D_t^\alpha x) - kx = 0,$$

This equation reduces to the equation of motion of the harmonic oscillator when $\alpha \rightarrow 1$.

If we consider the system of two planar pendula, both of length l and mass m , suspended a same distance apart on a horizontal line so that they moving in the same plane, the kinetic energy is

$$K = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2)$$

and the potential energy is

$$V = \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2),$$

where q_1 and q_2 denote the corresponding coordinates and g is the gravity constant (see Baleanu *et al.* (2012), for two-electric pendulum). The Lagrangian has the following form:

$$L(t, q_1, \dot{q}_1, q_2, \dot{q}_2) = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2).$$

The fractional form of this Lagrangian is given by

$$L(t, q_1, q_2, {}_a D_t^\alpha q_1, {}_a D_t^\alpha q_2) = \frac{1}{2} m [({}_a D_t^\alpha q_1)^2 + ({}_a D_t^\alpha q_2)^2] - \frac{1}{2} \frac{mg}{l} (q_1^2 + q_2^2).$$

To obtain Euler-Lagrange equations, we use

$$\frac{\partial L}{\partial q_i} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha q_i} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta q_i} = 0 \quad i = 1, 2.$$

It follows

$${}_t D_b^\alpha {}_a D_t^\alpha q_1 - \frac{g}{l} q_1 = 0, \quad {}_t D_b^\alpha {}_a D_t^\alpha q_2 - \frac{g}{l} q_2 = 0.$$

The classical Euler-Lagrange equations are obtained if $\alpha \rightarrow 1$:

$$\ddot{q}_1 + \frac{g}{l} q_1 = 0, \quad \ddot{q}_2 + \frac{g}{l} q_2 = 0.$$

As non-conservative system, we consider the damped harmonic oscillator. In this case, the frictional force F_f can be modeled as being proportional to the velocity v of the object, $F_f = -cv$, where c is the viscous damping coefficient. From

Newton's second law, it follows that

$$F = -kx - c \frac{dx}{dt} = m \frac{d^2x}{dt^2},$$

i.e.

$$m\ddot{x} + c\dot{x} + kx = 0.$$

The fractional Lagrangian and the fractional Rayleigh's dissipation function which describe this motion are

$$L(t, x, {}_a D_t^\alpha x) = \frac{1}{2} m ({}_a D_t^\alpha x)^2 - \frac{1}{2} k x^2 \text{ and } R = \frac{1}{2} ({}_a D_t^\alpha x)^2.$$

In this case,

$$F_f = -k {}_a D_t^\alpha x$$

and is derivable from fractional Rayleigh's dissipative function $R = \frac{1}{2} ({}_a D_t^\alpha x)^2$. We modify the standard fractional Euler-Lagrange equations by including the fractional Rayleigh's dissipation function with a time fractional derivative of the displacement. The fractional Euler-Lagrange equation takes the form

$$\frac{\partial L}{\partial x} + {}_t D_b^\alpha \frac{\partial L}{\partial {}_a D_t^\alpha x} + {}_a D_t^\beta \frac{\partial L}{\partial {}_t D_b^\beta x} - \frac{\partial R}{\partial {}_a D_t^\alpha x} = 0.$$

Substituting L and R in this equation we obtain

$$m {}_t D_b^\alpha x ({}_a D_t^\alpha x) - c_a (D_t^\alpha x) - kx = 0.$$

For $\alpha \rightarrow 1$ we get the equation of motion of the damped harmonic oscillator $m\ddot{x} + c\dot{x} + kx = 0$.

We can try to construct the classical mechanics related to the fractional calculus. We can introduce the fractional velocity $v(t)$ and fractional acceleration $a(t)$ as follows :

$$v(t) = {}_0^C D_t^\alpha x(t), \quad a(t) = {}_0^C D_t^\alpha v(t),$$

where ${}_0^C D_t^\alpha f(t)$ is the left Caputo fractional derivative (with fixed lower terminal 0 and moving upper terminal t)

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} \frac{d}{du} f(u) du, \quad 0 < \alpha < 1.$$

In the fractional mechanics, we can define Newton's equation by

$$F = ma = m {}_0^C D_t^\alpha v(t),$$

where m is a mass of the body. For a body in a resisting medium in which there exists a retarding force proportional to the fractional velocity in a uniform gravitational field, the equation of vertical motion is given by

$$m {}_0^C D_t^\alpha v = mg - kv, \quad v(0) = 0,$$

which is a fractional Cauchy problem. If we integrate this equation, we obtain

$$v(t) = g D_t^{-\alpha}(1) - \frac{k}{m} D_t^{-\alpha}(v(t)).$$

In order to find the formula of $v(t)$, we use a result from Saxena *et al.* (2010), which yields

$$v(t) = \frac{mg}{k} \left[1 - E_\alpha \left(-\frac{k}{m} t^\alpha \right) \right].$$

In the classical mechanics,

$$m\dot{v} = mg - kv, \quad v(0) = 0,$$

has the solution

$$v(t) = \frac{mg}{k} \left[1 - e^{-\frac{k}{m}t} \right].$$

We note that the terminal velocity is the same in both cases

$$\lim_{t \rightarrow \infty} v(t) = \frac{mg}{k}.$$

An extension of fractional Cauchy problem for general operators is given in Popescu (2010). Using a space-time fractional equation, in Popescu and Popescu (2010), the scaling and intermittent behavior of probability density functions of solar wind plasma parameters fluctuations is analyzed.

5. CONCLUSIONS

This paper intended to report some of the important results in the area of fractional calculus with applications to mechanics. Fractional differential models play a significant role in the description of the dynamics of many complex systems. It is presented an extension of variational calculus within the framework of fractional calculus. Fractional variational principles contain classical ones as a particular case when fractional operators converge to ordinary differential operators. Fractional mechanics, which is a non-local theory, describes both conservative and non-conservative systems. Using fractional derivatives and initial values of classical integer-order derivative with known physical interpretations, some illustrative applications on conservative and non-conservative systems were given.

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