AN EQUATION FOR ASTRONOMICAL DETERMINATION OF THE MOMENTS OF INERTIA OF THE EARTH

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Abstract. Using the properties of invariance of the moments of inertia with respect to the reduction of the equation of the ellipsoid of inertia to the canonical form, an equation for determination of the polar moment of inertia is established. Thus the number of equations for determination of the moments of inertia from the five harmonic coefficients of the second degree of the geopotential is equal to the number of unknowns. On the other hand, it is shown that the assumption of Erzhanov and Kalybaev concerning a certain relation between the moments of inertia about the axes of the geocentric coordinate system in which the series of the potential is given and the dynamical flattening is justified. New formulae for determination of the Eulerian angles concerning the orientation of the ellipsoid of inertia are obtained. The moments of inertia in the case of the Standard Earth II are computed.

Key words: geopotential - moments of inertia of the Earth - dynamical flattening.

1. INTRODUCTION

Theoretically, if the density distribution inside the Earth is a known function, its moments of inertia can be computed. The difficulty consists in the fact that this function is unknown, and it is necessary to make different assumptions about the Earth's internal constitution. A first approximation is obtained considering the Earth homogeneous. Improved values of the moments can be obtained considering the Earth constituted of homogeneous layers and the density a piecewise continuous and decreasing function from the center to the surface.

A possibility for a more precise determination of the moments of inertia of the Earth is offered by the series of the geopotential obtained by means of the Earth's artificial satellites. The five harmonic coefficients of the second degree, which are determined from observations, depend of the six unknown moments of inertia. Therefore, using these harmonic coefficients, one obtains five equations for the six moments of inertia. The system of equations would be complete if a new equation is added. Erzhanov and Kalybaev (1975, 1984) proposed the equation $[C' - (A' + B')/2]/C' = H$, where the dynamical flattening H is obtained from the constant of lunisolar precession and A', B', C' are the unknown moments of inertia with respect to the geocentric coordinate system in which the series is given. But, because $H = [C-(A+B)/2]/C$, A, B, C being the principal moments of inertia, it is necessary to prove that $[C'-(A'+B')/2]/C'$ can be replaced by H. In the present paper, this assumption is proved. For this purpose, using the properties of invariance of the moments with respect to the reduction of the equation of the ellipsoid of inertia to the canonical form, an equation for determination of the polar moment is established. Then new formulae concerning the orientation of the ellipsoid of inertia are obtained, and the moments of inertia for the Standard Earth II are computed.

2. THE GEOPOTENTIAL AND THE MOMENTS OF INERTIA

In the geocentric coordinate system $O\xi\eta\zeta$, in which the axis $O\zeta$ is oriented toward the Conventional International Origin, and the plane $O\xi\zeta$ is the origin plane for longitude (Greenwich meridian plane), the gravitational potential of the Earth is given by the series

$$\begin{aligned} U(r,\varphi,\lambda) &= \frac{GM}{r}[1-\sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n J_n P_n(\sin\varphi) \\ &+\sum_{n=2}^{\infty}\sum_{k=1}^{n}\left(\frac{a_e}{r}\right)^n P_n^{(k)}(\sin\varphi)(C_{nk}\cos k\lambda+S_{nk}\sin k\lambda)], \end{aligned} \tag{1}$$

where G is the gravitational constant, M and a_e are the mass and the equatorial radius of the Earth, P_n is the polynomial of Legendre of n^{th} degree, and $P_n^{(k)}$ is the associated function of Legendre of n^{th} degree and k^{th} order. On the other hand, r, λ, φ are the polar coordinates, r being the geocentric distance, λ the longitude and φ the geocentric latitude.

The harmonic coefficients of the second degree can be expressed in function of the moments of inertia. One obtains the following relations

$$\begin{aligned} Ma_e^2 J_2 &= C' - \frac{A'+B'}{2}, \\ 4Ma_e^2 C_{22} &= B'-A', \\ Ma_e^2 C_{21} &= E', \\ Ma_e^2 S_{21} &= D', \\ 2Ma_e^2 S_{22} &= F'. \end{aligned} \tag{2}$$

These five relations and the relation proposed by Erzhanov and Kalybaev,

$$H = \frac{C'-(A'+B')/2}{C'}, \tag{3}$$

allow the determination of the moments A', B', C', D', E', F'. For the moments of inertia in the system $O\xi\eta\zeta$ we keep the notations utilized by Erzhanov and Kalybaev (1984), namely

$$\begin{aligned} A' &= \int_V \rho(\xi,\eta,\zeta)(\eta^2+\zeta^2)\mathrm{d}v, \\ B' &= \int_V \rho(\xi,\eta,\zeta)(\xi^2+\zeta^2)\mathrm{d}v, \\ C' &= \int_V \rho(\xi,\eta,\zeta)(\xi^2+\eta^2)\mathrm{d}v, \\ D' &= \int_V \rho(\xi,\eta,\zeta)\zeta\eta\mathrm{d}v, \\ E' &= \int_V \rho(\xi,\eta,\zeta)\zeta\xi\mathrm{d}v, \\ F' &= \int_V \rho(\xi,\eta,\zeta)\xi\eta\mathrm{d}v, \end{aligned} \tag{4}$$

where V is the domain occupied by the Earth and ρ is the density.

The presented form of the geopotential has been recommended by the International Astronomical Union (Hagihara, 1962). But in the papers consecrated to the determination of the coefficients of the geopotential is frequently used the following form (Aksenov, 1977)

$$\begin{aligned} U &= \frac{GM}{r}[1-\sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^n J_n P_n(\sin\varphi) \\ &\quad +\sum_{n=2}^{\infty}\sum_{k=1}^{n}\left(\frac{a_e}{r}\right)^n p_n^{(k)}(\sin\varphi)(\overline{C}_{nk}\cos k\lambda+\overline{S}_{nk}\sin k\lambda)], \end{aligned} \tag{5}$$

where $p_n^{(k)}$ is the fully normalized associated function of Legendre, and the coefficients $\overline{C}_{nk}$ and $\overline{S}_{nk}$ have the expressions

$$\begin{aligned} \overline{C}_{nk} &= \sqrt{\frac{(n+k)!}{2(n-k)!}}\frac{C_{nk}}{\sqrt{2n+1}}, \\ \overline{S}_{nk} &= \sqrt{\frac{(n+k)!}{2(n-k)!}}\frac{S_{nk}}{\sqrt{2n+1}}. \end{aligned} \tag{6}$$

We mention that the geopotential can be also represented by another series (Erzhanov and Kalybaev, 1984; Mueller, 1964).

For the normalized coefficients of the second degree one obtains

$$\begin{aligned} \overline{C}_{21} &= \sqrt{\frac{3}{5}}C_{21},\ \overline{S}_{21}=\sqrt{\frac{3}{5}}S_{21}, \\ \overline{C}_{22} &= \sqrt{\frac{12}{5}}C_{22},\ \overline{S}_{22}=\sqrt{\frac{12}{5}}S_{22}. \end{aligned} \tag{7}$$

3. AN EQUATION FOR DETERMINATION OF THE POLAR MOMENT OF INERTIA

The five relations (2) are insufficient to determine the six moments of inertia. For this reason it is necessary to deduce another equation.

In the geocentric coordinate system $O\xi\eta\zeta$, the equation of ellipsoid of inertia is

$$A'\xi^2+B'\eta^2+C'\zeta^2-2D'\eta\zeta-2E'\xi\zeta-2F'\xi\eta=1. \tag{8}$$

In the system $Oxyz$, defined by the principal axes of inertia, the equation becomes

$$Ax^2+By^2+Cz^2=1, \tag{9}$$

A, B, C being the principal moments of inertia, which can be determined by solving the secular equation

$$\begin{vmatrix} A'-q & -F' & -E' \\ -F' & B'-q & -D' \\ -E' & -D' & C'-q \end{vmatrix}=0 \tag{10}$$

or

$$\begin{aligned} & q^3-(A'+B'+C')q^2+ \\ & +(A'B'+A'C'+B'C'-D'^2-E'^2-F'^2)q- \\ & -(A'B'C'-2D'E'F'-A'D'^2-B'E'^2-C'F'^2)=0. \end{aligned} \tag{11}$$

Because the values A, B, C of the principal moments of inertia do not depend on the orientation of the system $O\xi\eta\zeta$, it results that the coefficients of the secular equation are invariant. Therefore we can write

$$\begin{aligned} A+B+C &= A'+B'+C', \\ AB+AC+BC &= A'B'+A'C'+B'C'-D'^2-E'^2-F'^2, \\ ABC &= A'B'C'-2D'E'F'-A'D'^2-B'E'^2-C'F'^2. \end{aligned} \tag{12}$$

From the expression of the dynamical flattening

$$H = \frac{C-(A+B)/2}{C} \tag{13}$$

one obtains

$$C=\frac{A+B}{2(1-H)}. \tag{14}$$

On the other hand, taking into account the relations

$$\begin{aligned} A+B+C &= A'+B'+C', \\ Ma_e^2J_2 &= C'-(A'+B')/2, \end{aligned}$$

we obtain

$$C=\frac{3C'-2J_2Ma_e^2}{3-2H}. \tag{15}$$

From the relations $AB+AC+BC=AB+(A+B)C=h$, $ABC=k$, where h and k are the values of the corresponding invariants, we can write $AB=k/C$ and $k/C+(A+B)C=h$, and therefore $A+B=h/C-k/C^2$. Substituting this expression of A+B in the expression (14) of C, it results

$$2(1-H)C^3-hC+k=0 \tag{16}$$

or

$$\begin{aligned} 2(1 \quad - \quad H)&\left(\frac{3C'-2J_2Ma_e^2}{3-2H}\right)^3 \\ &-\frac{3C'-2J_2Ma_e^2}{3-2H}(A'B'+A'C'+B'C'-D'^2-E'^2-F'^2) \\ &+A'B'C'-2D'E'F'-A'D'^2-B'E'^2-C'F'^2=0. \end{aligned} \tag{17}$$

But, from the relations $A'+B'=2C'-2J_2Ma_e^2=2C'+a'$, $B'-A'=4Ma_e^2C_{22}=b'$, where $a'=-2J_2Ma_e^2$ and $b'=4Ma_e^2C_{22}$ are known, one obtains

$$\begin{aligned} A' &= C'+\frac{a'-b'}{2}, \\ B' &= C'+\frac{a'+b'}{2}. \end{aligned} \tag{18}$$

Substituting in the last equation, this becomes

$$ax^3+bx^2+cx+d=0, \tag{19}$$

where

$$\begin{aligned}
x &= C', \\
a &= 8H^3, \\
b &= 8H^3 a', \\
c &= -2H(3-4H)a'^2 + 2H(3-2H)^2\left(\frac{a'^2-b'^2}{4} - D'^2 - E'^2 - F'^2\right), \\
d &= -2(1-H)a'^3 + (3-2H)^2 a'\left(\frac{a'^2-b'^2}{4} - D'^2 - E'^2 - F'^2\right) + \\
&\quad +(3-2H)^3\left(2D'E'F' + \frac{a'+b'}{2}E'^2 + \frac{a'-b'}{2}D'^2\right).
\end{aligned} \tag{20}$$

Because for the models of geopotential $a > 0$, $b < 0$, $c < 0$, $d < 0$, according to Descartes' rule of signs, the obtained equation (19) has only one positive root.

The system of six independent equations (2) and (19) allows the determination of the moments of inertia A', B', C', D', E', F'. On the other hand, solving the secular equation one obtains the principal moments A, B, C.

Because the Earth's mass is determined with a less accuracy, it is preferably to use the normalized moments of inertia $\overline{A}' = A'/Ma_e^2$, etc., and the normalized values $\overline{a}' = a'/Ma_e^2 = -2J_2$, $\overline{b}' = b'/Ma_e^2 = 4C_{22}$. In this case, the equation (19) becomes

$$\bar{a}\bar{x}^3 + \bar{b}\bar{x}^2 + \bar{c}\bar{x} + \bar{d} = 0, \tag{21}$$

where

$$\begin{aligned}
\overline{x} &= \overline{C}', \\
\overline{a} &= 8H^3, \\
\overline{b} &= 8H^3 \overline{a}', \\
\overline{c} &= -2H(3-4H)\overline{a}'^2 + 2H(3-2H)^2\left(\frac{\overline{a}'^2-\overline{b}'^2}{4} - \overline{D}'^2 - \overline{E}'^2 - \overline{F}'^2\right), \\
\overline{d} &= -2(1-H)\overline{a}'^3 + (3-2H)^2 \overline{a}'\left(\frac{\overline{a}'^2-\overline{b}'^2}{4} - \overline{D}'^2 - \overline{E}'^2 - \overline{F}'^2\right) + \\
&\quad +(3-2H)^3\left(2\overline{D}'\overline{E}'\overline{F}' + \frac{\overline{a}'+\overline{b}'}{2}\overline{E}'^2 + \frac{\overline{a}'-\overline{b}'}{2}\overline{D}'^2\right).
\end{aligned} \tag{22}$$

Evidently, $\overline{H} = [\overline{C} - (\overline{A} + \overline{B})/2]/\overline{C} = H$. The equations (2) become

$$\begin{aligned}
\overline{C}' - \frac{(\overline{A}' + \overline{B}')}{2} &= J_2, \\
\overline{B}' - \overline{A}' &= 4C_{22}, \\
\overline{D}' &= S_{21}, \\
\overline{E}' &= C_{21}, \\
\overline{F}' &= 2S_{22}.
\end{aligned} \tag{23}$$

On the other hand, in the case of the normalized moments the secular equation has the same form and give the normalized principal moments $\overline{A}$, $\overline{B}$, $\overline{C}$.

We mention that the value of C' obtained by means of the deduced equation (Vîlcu, 2009) practically coincides with the value obtained taking into account the relation $[C' - (A' + B')/2]/C' = H$. Thus the relation proposed by Erzhanov and Kalybaev is justified. Evidently, because the moments D', E', F' are very small in comparison with the moments A', B', C', practically $[C' - (A' + B')/2]/C' = [C - (A + B)/2]/C = H$. But in the general case it is necessary to use the obtained equation for the polar moment.

4. ORIENTATION OF THE ELLIPSOID OF INERTIA

The orientation of the ellipsoid of inertia, *i.e.* the orientation of the principal axes of inertia with respect to the system $O\xi\eta\zeta$ can be given either by spherical coordinates (longitude and geocentric latitude) or by the Eulerian angles. We shall use the second possibility, as in the work of Erzhanov and Kalybaev (1984), but in a different manner. Let $\beta = (\widehat{O\xi, ON})$, $\alpha = (\widehat{ON, Ox})$, $\gamma = (\widehat{O\zeta, Oz})$ be the Eulerian angles, ON being the line of intersection between the planes $O\xi\eta$ and Oxy.

Let $(\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$ $(i = 1, 2, 3)$ be the direction cosines of the principal axes of inertia Ox, Oy, Oz with respect to $O\xi\eta\zeta$. They have the following expressions

$$
\begin{aligned}
\gamma_{11} &= \cos\beta\cos\alpha - \sin\beta\sin\alpha\cos\gamma, \\
\gamma_{12} &= \sin\beta\cos\alpha + \cos\beta\sin\alpha\cos\gamma, \\
\gamma_{13} &= \sin\alpha\sin\gamma, \\
\gamma_{21} &= -\cos\beta\sin\alpha - \sin\beta\cos\alpha\cos\gamma, \\
\gamma_{22} &= -\sin\beta\sin\alpha + \cos\beta\cos\alpha\cos\gamma, \\
\gamma_{23} &= \cos\alpha\sin\gamma, \\
\gamma_{31} &= \sin\beta\sin\gamma, \\
\gamma_{32} &= -\cos\beta\sin\gamma, \\
\gamma_{33} &= \cos\gamma,
\end{aligned} \tag{24}
$$

and are given by the relations

$$
\frac{\gamma_{i1}}{\delta_{i1}} = \frac{\gamma_{i2}}{\delta_{i2}} = \frac{\gamma_{i3}}{\delta_{i3}} = \frac{1}{(\delta_{i1}^2 + \delta_{i2}^2 + \delta_{i3}^2)^{1/2}}, \tag{25}
$$

where $\delta_{i1}, \delta_{i2}, \delta_{i3}$ (i=1,2,3) are the cofactors of the elements in the row i of the determinant Δ which appears in the secular equation, q being successively replaced by A, B, C. On the other hand, the system $Oxyz$ has the same orientation as the system

$O\xi\eta\zeta$, if the following condition is fulfilled (Efimov, 1972)

$$\begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} = 1. \tag{26}$$

Once determined the direction cosines, one can obtain the Eulerian angles by the formulae

$$\begin{aligned} \cos\gamma &= \gamma_{33}, \\ \sin\alpha\sin\gamma &= \gamma_{13}, \\ \cos\alpha\cos\gamma &= \gamma_{23}, \\ \sin\beta\sin\gamma &= \gamma_{12}\gamma_{23} - \gamma_{22}\gamma_{13}, \\ \cos\beta\sin\gamma &= \gamma_{11}\gamma_{23} - \gamma_{21}\gamma_{13}. \end{aligned} \tag{27}$$

We mention that analogous formulae are used for determination of the angular elements in the methods of orbit determination.

If $\gamma = 0$, then the plane Oxy coincides with $O\xi\eta$, and the orientation of the ellipsoid is reduced to the orientation of the ellipse of inertia. From (24) it is results

$$\begin{aligned} \gamma_{11} &= \cos(\beta+\alpha) = \cos\lambda, \\ \gamma_{12} &= \sin(\beta+\alpha) = \sin\lambda, \\ \gamma_{21} &= -\sin(\beta+\alpha) = -\sin\lambda, \\ \gamma_{22} &= \cos(\beta+\alpha) = \cos\lambda, \end{aligned} \tag{28}$$

where λ is the longitude of the axis Ox in the plane $O\xi\eta$.

Evidently, if the direction cosines are determined, then the orientation of the principal axes can be also given by spherical coordinates, as in the case of velocity ellipsoid of the stars (Mihaila, 1974).

5. STANDARD EARTH II

For to give an example, we present the results obtained for the geopotential model concerning the Standard Earth II (Gaposhkin and Lambeck, 1970), a model analyzed by Erzhanov and Kalybaev. For this model the geocentric gravitational constant $GM = 3.986013 \times 10^{-14}\mathrm{m}^3\mathrm{s}^{-2}$ and the equatorial radius $a_e = 6378155$ m. On the other hand, the harmonic coefficients of the second degree have the values $J_2 = 1082.628 \times 10^{-6}$, $\overline{C}_{21} = \overline{S}_{21} = 0$, $\overline{C}_{22} = 2.41290 \times 10^{-6}$, $\overline{S}_{22} = -1.36410 \times 10^{-6}$.

We shall use the value of the dynamical flattening given by Petit and Luzum (2010), namely $H = 0.003273795$. One obtains for the normalized moments of inertia

$$\begin{aligned} \overline{A}' &= 0.329609368, \\ \overline{B}' &= 0.329615598, \\ \overline{C}' &= 0.330695111, \\ \overline{D}' &= \overline{E}' = 0, \\ \overline{F}' &= -1.761045528 \times 10^{-6}. \end{aligned} \tag{29}$$

Solving the secular equation, it results for the normalized principal moments

$$\begin{aligned} \overline{A} &= 0.329608907, \\ \overline{B} &= 0.329616059, \\ \overline{C} &= 0.330695111. \end{aligned} \tag{30}$$

Using the value of the gravitational constant $G = 6.67428 \times 10^{-11} m^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$, adopted in the IAU(2009) System of astronomical constants, we obtain for the principal moments of inertia (in $10^{37} \mathrm{kg} \cdot \mathrm{m}^2$)

$$\begin{aligned} A &= 8.007187118, \\ B &= 8.008160879, \\ C &= 8.034376902. \end{aligned} \tag{31}$$

The values of the moments are given with nine or ten significant digits only to obtain H with seven significant digits. We note anew that the value of the polar moment C' obtained by means of the deduced equation practically coincides with the value given by the Erzhanov and Kalybaev relation, the difference being at most of the order of 10^{-9} for the normalized moment.

In the case of the Standard Earth II, because the angle $\gamma = 0$, the orientation of the ellipsoid of inertia is reduced to the orientation of the ellipse of inertia in the equatorial plane. One obtains for the longitude of the axis corresponding to the minimum moment (A) the value $\lambda = -14.7^0$. The obtained results are slightly different from the results obtained by Erzhanov and Kalybaev (1984), who used the value $H = 0.00327364$ for the dynamical flattening.

We mention that the equation was already used for different models of geopotential (Vîlcu, 2009).

6. CONCLUSION

In the present paper an equation for determination of the polar moment of inertia of the Earth is obtained. The equation contains the dynamical flattening, which are deduced from the constant of lunisolar precession. According to Descartes' rule of signs, this cubic equation has only one positive root. The value of the normalized polar moment deduced by solving the equation practically coincides with the value obtained taking into account the relation proposed by Erzhanov and Kalybaev, the difference being at most of the order of 10^{-9}. Therefore their assumption is justified.

The cause consists in the fact that the product of inertia (D', E', F') are very small in comparison with the moments about the axes (A', B', C'). We mention that the obtained equation is exact not only in this case, but also in the general case. On the other hand, the given formulae regarding the orientation of the ellipsoid of inertia allow the univocal determination of the Eulerian angles.

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