A METHOD FOR EXOPLANETARY SYSTEMS ELEMENTS DETERMINATION

ALEXANDRU DUMITRESCU, STEFAN GABRIEL SORESCU-SURDU

Astronomical Institute of the Romanian Academy
Str. Cutitul de Argint 5, RO-75212, Bucharest 28, Romania
E-mail: alex@aira.astro.ro; sorescus@yahoo.com

Abstract: The light curves of exoplanetary system transits show only one minimum. The lack of the secondary minima makes difficult the determination of the orbital elements for elliptic orbits. For circular orbits we determine the intervals for radius ratio, the interior tangent angle and limb darkening of the star. For each (k, θ_i) we compute the star radius and orbital inclination. For elliptic orbits the eccentricity and periastron longitude as function of transit time and light curve asymmetry are determinated.

Key words: stellar photometry, exoplanetary systems.

1. INTRODUCTION

The first discoveries of exoplanets were made by photometric observations. The light curves of the parent stars show periodical variations of small amplitude (about hundredths magnitudes). The lack of secondary minima shows that the observed phenomena are due to the transit of a small size body. The hypothesis of a cold dwarf star companion was rejected because the light curves show no observable secondary minima (occultation). So, the companion of the parent star was considered to be a planet. In order to produce a detectable drop of magnitude the planet must have a big size, similar with giant planets of our solar system. Several observational techniques were developed and proved the existence of exoplanetary systems.

The principal techniques used in present are:

– direct imaging of the star-planet system
– interferometric imaging of the star-planet system
– detection of the planetary system in a composite spectrum of star and planet
– astrometric detection of the star motion around the star-planet center of mass

– radial velocity measurement of the parent star.

The detection of the Earth-like planets by photometrical methods is more difficult because of the small effects produced by transit.

2. ORBITAL ELEMENTS CALCULATION

2.1. THE CASE OF CIRCULAR ORBITS

We will use the following symbols:

r_s – star relative radius
r_p – planet relative radius
$k = r_p/r_s$
a – mean separation
i – orbit inclination
l_0 – normalized light at minimum time
u – limb darkening of the star
θ − orbital angle relative to the observed light minimum
θ_e – first contact angle
θ_i – second contact angle
Δ – projection of the star – planet separation

For $u = 0$ we have the following equation:

$$k^2 = 1 - l_0 . \tag{1}$$

For $u \neq 0$ we have:

$$k^2 \cong \frac{(3-u)(1-l_0)}{3(1-u+u\cos\varphi_0)} \tag{2}$$

where

$$\cos\varphi_0 = \sqrt{1 - \frac{a^2\cos^2 i}{r_s^2}} . \tag{3}$$

In equation (2) one considered that the brightness is constant for the entire stellar surface occulted by the planet at one given moment. In the case of central transit we have:

$$k_{\text{inf}} = \left[\frac{(3-u)(1-l_0)}{3} \right]^{1/2}. \tag{4}$$

This value represents the inferior limit for k.

For general case (non-central transit) we have

$$k > \left[\frac{(3-u)(1-l_0)}{3} \right]^{1/2}. \tag{5}$$

In the case of grazing annular transit ($\varphi \approx 90^0$) from relation (2) one obtains the superior limit for k,

$$k_{\text{sup}} \approx \left[\frac{(3-u)(1-l_0)}{3(1-u)} \right]^{1/2}. \tag{6}$$

From the light curve we can determine the depth of the minima ($1–l_0$) and exterior tangent angle θ_e with sufficient accuracy.

From the theory of the spherical eclipsing binary stars we have:

$$\Delta = r_s + p\,r_p = r_s(1 + k\,p) \tag{7}$$

where the value of p lies between 1 and $1/r_p$ during the transit.

Since

$$\Delta^2 = a^2(\sin^2\theta + \cos^2\theta\cos^2 i) \tag{8}$$

with $a = 1$, for the parameter p we will obtain the value

$$p = \frac{1}{r_p}\sqrt{\cos^2 i + \sin^2 i \sin^2\theta} - \frac{1}{k}. \tag{9}$$

For $\theta = \theta_e$, $p = 1$ and $\theta = \theta_i$, $p = –1$. By replacing these values of p in equation (8) one obtains the following equations:

$$1 + \frac{1}{k} = \frac{1}{r_p}\sqrt{\cos^2 i + \sin^2 i \sin^2\theta_e} \tag{10a}$$

$$-1 + \frac{1}{k} = \frac{1}{r_p}\sqrt{\cos^2 i + \sin^2 i \sin^2\theta_i}. \tag{10b}$$

From equations (10a) and (10b):

$$b = \left(\frac{1+k}{1-k}\right)^2 = \frac{\cos^2 i + \sin^2 i \sin^2 \theta_e}{\cos^2 i + \sin^2 i \sin^2 \theta_i}. \tag{11}$$

From relation (11) results:

$$\tan^2 i = \frac{1-b}{b\sin^2 \theta_i - \sin^2 \theta_e}. \tag{12}$$

Since b>1, from (12) one results the condition:

$$b\sin^2 \theta_i - \sin^2 \theta_e < 0 .$$

We find by this way the superior limit of θ_i

$$\begin{aligned} \sin\theta_i &< \frac{\sin\theta_e}{\sqrt{b}} = \frac{\sin\theta_e(1-k)}{1+k} \\ \theta_{i\ \mathrm{sup}} &= \arcsin\frac{\sin\theta_e(1-k)}{1+k}. \end{aligned} \tag{13}$$

For determining the value of i we will do successive trials. From stellar models we choose the limb darkening coefficient. From the light curve we know $1 - l_0$ and θ_e. For each value of k ($k_{\mathrm{inf}} < k < k_{\mathrm{sup}}$) we find from equation (13) the possible values of θ_i in the domain $[0, \theta_{i\ \mathrm{sup}}]$. For each pair ($k$, θ_i) from equation (12) one obtains the value of i.

By equation (10) we get the relative radius of the planet:

$$r_p = k\frac{\sqrt{\cos^2 i + \sin^2 i \sin^2 \theta_e}}{k+1}. \tag{14}$$

The relative star radius will be:

$$r_s = \frac{r_p}{k}.$$

For each set of elements (k, r_p, i, u) we can calculate the theoretical light-curve.

Observation of planetary transits allows determining with great accuracy the value of limb darkening coefficient of the star. For eclipsing binary stars the light-curves show different phenomena such as non-spherical components, mass transfer, and existence of spots, intrinsic variability, and the existence of circumstellar matter and so on. In such a case is difficult to determine the value of u with high accuracy. For planetary transits the light-curves are very smooth, all the factors which distort the light-curves missing.

For small values of r_s/a (large orbits) and for $i \neq 90°$, the projection of the orbital arcs between the exterior contacts can be approximated by a straight line. In this case the total duration of the transit (t_{tr}) and the duration of annular phase (t_{ann}) are given by:

$$t_{tr} = \frac{P}{\pi}\sqrt{r_s(1+k)^2 - \cos^2 i} \tag{15}$$

$$t_{ann} = \frac{P}{\pi}\sqrt{r_s^2(1-k)^2 - \cos^2 i}\ . \tag{16}$$

The values of P and t_{tr} can be accurately determined from observations. For different values of t_{ann}, close to the point in which the slope of the curve shows a significant change, and with the value of u obtained from stellar models, from equations (2), (3), (15), (16) we can obtain the approximate the values of k, r_s and i.

2.2. THE CASE OF ELLIPTICAL ORBITS

Many exoplanet orbits show eccentricity. Statistically, when the larger semi-major axis is bigger than the eccentricity is higher. For such cases the light curve is asymmetrical and the time of minima do not correspond with the inferior conjunction of the planet for $\omega \neq 90°$ and 270°. Let's consider the star fixed and υ the longitude related to inferior conjunction time in orbital plane of the planet's. We have:

$$v = 270^o + \upsilon - \omega \tag{17}$$

where v is the true anomaly of the planet.

In this case:

$$\Delta = \frac{a(1-e^2)}{1+e\sin(\upsilon-\omega)}\sqrt{1-\sin^2 i\cos^2\upsilon}\ . \tag{18}$$

The light minimum will be for:

$$\upsilon_0 = hctg^2 i(1+g\cos ec^2 i) \tag{19}$$

(Martinov 1971), where:

$$\begin{aligned} h &= e\cos\omega \\ g &= e\sin\omega. \end{aligned} \tag{20}$$

Mean anomaly expressed in radians can be approximated with the following relation:

$$M = \upsilon - 2e\sin\upsilon + \frac{3}{4}e^2\sin 2\upsilon - ... \tag{21}$$

From (17) and (21) result:

$$M = \frac{2\pi}{3} + \upsilon - \omega + 2e\cos(\upsilon - \omega) - \frac{3}{4}e^2\sin 2(\upsilon - \omega)\,. \tag{22}$$

We note by υ_1 (egress) and υ_2 (ingress) ($\upsilon_1 < \upsilon_2$) the planet orbital longitudes for exterior contacts and by M_1 and M_2 the corresponding mean anomalies. If M_0 is the mean anomaly for the observed photometric minimum

$$\begin{aligned} M_1 - M_0 = \upsilon_1 - \upsilon_0 + 2e\cos(\upsilon_1 - \omega) - \frac{3}{4}e^2\sin 2(\upsilon_1 - \omega) - 2e\cos(\upsilon_0 - \omega) + \\ \frac{3}{4}e^2\sin 2(\upsilon_0 - \omega) = \frac{\pi}{180^0}\theta_1 \end{aligned} \tag{23}$$

$$\begin{aligned} M_2 - M_2 = \upsilon_2 - \upsilon_1 - 2e\cos(\upsilon_1 - \omega) + \frac{3}{4}e^2\sin 2(\upsilon_1 - \omega) + 2e\cos(\upsilon_2 - \omega) - \\ \frac{3}{4}e^2\sin 2(\upsilon_2 - \omega) = \frac{\pi}{180^0}(\theta_2 - \theta_1) \end{aligned} \tag{24}$$

where θ_1 and θ_2 are the observed exterior tangent angles.

The maximum asymmetry of the light curve is reached for $\omega = 0$ and $\omega = \pi$. The value of $|\upsilon_0|$ shows a relative maximum for $\omega = 0$ and $\omega = \pi$. As smaller is *I*, as greater is the value of $|\upsilon_0|$. For example for $\omega = 0$, e = 0.50 and $i = 80°$ from equation (19) results $\upsilon_0 = 0.88°$. The mean anomaly corresponding to υ_0 is:

$$M_0 = \frac{2\pi}{3} + \upsilon_0 - \omega + 2e\cos(\upsilon_0 - \omega) - \frac{3}{4}e^2\sin(2(\upsilon_0 - \omega))\,. \tag{25}$$

The value of θ will be:

$$\theta = M - M_0\,. \tag{26}$$

Now, we can compare the theoretical orbit with the observed one.

To find the orbital elements we proceed as follows: the values of υ_0, υ_1, υ_2 there are in a few degrees interval centered on 0, θ_1 and θ_2 , respectively. For each set of (υ_0, υ_1, υ_2) from equations (23) and (24) we find the values for e and ω, where the values of θ_1 and θ_2 are obtained from the observed light curve. The i is given by eq.(19). Relation (18) one can write for υ_1:

$$r_s\ (1+k)=\frac{a(1-e^2)}{1+e\sin(\upsilon_1-\omega)}\sqrt{1-\sin^2 i\cos^2\upsilon_1}\ . \tag{27}$$

With k approximated by (2) one obtains r_s. With these elements one can calculate the theoretical light curve $l_{th}(\upsilon)$. To compare the $l_{th}\,(\upsilon)$ with the observed light curve we have to express θ as function of υ. By (23) one gets:

$$\theta=\frac{180^o}{\pi}\begin{pmatrix}\upsilon-\upsilon_0+2e\cos(\upsilon-\omega)-\frac{3}{4}e^2\sin 2(\upsilon-\omega)-2e\cos(\upsilon-\omega)+\\ \frac{3}{4}e^2\sin 2(\upsilon_0-\omega)\end{pmatrix}. \tag{28}$$

By successive approximations for different υ_0, υ_1, υ_2 we choose the theoretical light curve $l_{th}(\theta)$ that provides the best fitting with the observed light curve $l_{obs}(\theta)$. It's obvious that υ_0, $\upsilon_1$and υ_2 there are close to 0, θ_1 and θ_2 respectively. We mention that for $\upsilon_0 > 0$, ω lies between $-\pi/2$ and $\pi/2$ and for $\upsilon_0 < 0$, ω lies between $\pi/2$ and $3\pi/2$.

During the transit the orbit can be considered as a circular one with mean radius:

$$a' = \frac{a(1-e^2)}{1+e\sin^*(\upsilon-\omega)} \tag{29}$$

where $\sin^*(\upsilon-\omega)$ is the mean value given by:

$$\eta=\sin^*(\upsilon-\omega)=\frac{\int\sin(\upsilon-\omega)d(\upsilon-\omega)}{\int d(\upsilon-\omega)}\ . \tag{30}$$

Denoting with k', r_g' and i' the values as function of a' obtained for elliptical orbit one can find the correspondent values taking a as unit:

$$k = k^{'}$$
$$r_g = r^{'}{}_g /(1 + g\eta) \tag{31}$$
$$\tan i = \tan i^{'}(1 + 2g\eta)$$

(Martinov, 1971).

3. CONCLUSIONS

The most approachable method to observe the exoplanet transits is the photometric one. A good seeing and a high quality receiver would permit the photometric observations even with small instruments. The precision of photometric observations is ultimately limited by photon noise and by the sky background. These errors can be reduced by increasing the exposure time T, because they scale with $T^{-1/2}$.

In case of giant planets the light reflected by planet provides a small light variation outside of transit depending on albedo, phase angle (α) and wavelength. The intensity ratio between the planetary and stellar emission is:

$$\varepsilon_\lambda(\alpha) = p_\lambda (r_p / a)^2 \phi_\lambda(\alpha) \tag{32}$$

where p_λ is the geometric albedo and $\Phi(\alpha)$ is the phase function.

In case of the a Lambert sphere, which scatters all incoming photons isotropic, p_λ=2/3 and

$$\phi_\lambda(\alpha) = \frac{\sin\alpha + (\pi - \alpha)\cos\alpha}{\pi}\,. \tag{33}$$

With the precision of the photometric space missions, it should thus be possible to detect the starlight reflected by *hot Jupiters* through their phase variation. At the end of transit and beginning of occultation the phases of the planet are very close to 0 and 1, respectively, even for small orbits. As example, with the elements of the system HD209458 (Brown *et al.*, 2001) the magnitude variation between transit and occultation should be Δm = 0.0014 in case of the Lambert sphere. From (32) one can estimate p_λ taking $\Phi_\lambda(0) = 1$

$$p_\lambda = (2.512^{\Delta m} - 1)(a / r)^2\,. \tag{34}$$

A statistical study about the exoplanet orbits (Santos *et al.*, 2003) pointed out the followings:

– All planets with short periods ($P < 10$ days) have small eccentricities ($e < 0.1$).

– Eccentricity-period relations for exoplanet and binary stars are very similar.

– There is a exoplanet class with long periods having almost circular orbits.

REFERENCES

Martinov, D.I.: 1971, "*Zatmenie Peremenie Zvezdi*", 313, Moscva.

Santos, N.C., Udry, S., Mayor, M. *et al.*: 2003, A&A, **406**, 373.

Brown, T.M., Charbonneau, D., Gilliland, R.L., Noyes, R.W., Burrow, A.: 2001, "*Hubble Space Telescope Time Series Photometry of the Transiting Planet of HD 209458"*, Ap.J., **552**, 699–709.

Received on 2 August 2012