

REGULARIZATION OF THE CIRCULAR RESTRICTED THREE-BODY PROBLEM USING “SIMILAR” COORDINATE SYSTEMS

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Abstract. The regularization of a new problem, namely the three-body problem using “similar” coordinate system, is proposed. First we write the Hamiltonian function, the equations of motion in canonical form, then, using the generating function, we construct the transformed equations of motion. After the coordinate transformations we introduce the fictitious time, to regularize the equations of motion. Explicit formulas are given for the regularization. Using the resulted regularized equations, we analyze these canonical equations numerically, for the Earth-Moon binary system.

Key words: circular restricted three-body problem – regularization – similar coordinate system.

1. INTRODUCTION

The regularization method has become popular in recent years (Waldvogel 2006; Celletti et al. 2011) for long term studies of the motion of celestial bodies. These problems have a special merit in astronomy, because with their help we can study with more efficiency the equations of motion with singularities. At the collision the equations of motion possess singularities. The problem of singularities plays an important role under computational, physical and conceptual aspects (Csillik 2003). The singularities occurring at collisions can be eliminated by the proper choice of the independent variable. The basic idea of regularization procedure is to compensate for the infinite increase of the velocity at collision. For this reason, a new independent variable, fictitious time, is adopted. The corresponding equations of motion are regularized by two transformations: the time transformation and the coordinate transformation. The most important part of the regularization is the time transformation. We use a new fictitious time to slow the motion near the singularities. In the following we describe the circular restricted three-body problem (CRTBP), and the circular restricted three-body problem using the similar

coordinate system (CRSTBP).

According to Kopal (1978) or Roman (2010), the scalar equations of motion for the circular restricted three-body problem (CRTBP) in eclipsing binary systems are

CASE 1: In the $(q_1 O_1 q_2 q_3)$ coordinate system the equations of motion (in usual notation) are

$$\begin{aligned} d^2 q_1 / dt^2 - 2dq_2 / dt &= q_1 - \frac{q}{1+q} - \frac{q_1}{(1+q)r_1^3} - \frac{q(q_1-1)}{(1+q)r_2^3}, \\ d^2 q_2 / dt^2 - 2dq_1 / dt &= q_2 - \frac{q}{1+q} - \frac{q_2}{(1+q)r_1^3} - \frac{qq_2}{(1+q)r_2^3}, \\ d^2 q_3 / dt^2 &= -\frac{q_3}{(1+q)r_1^3} - \frac{qq_3}{(1+q)r_2^3}, \end{aligned} \quad (1)$$

where $r_1 = \sqrt{q_1^2 + q_2^2 + q_3^2}$, $r_2 = \sqrt{(q_1-1)^2 + q_2^2 + q_3^2}$. We choose as mass unit the sum of the masses of the components of the binary system, for the distance between the centers of the components. The Hamiltonian of (CRTBP) system (1) takes the form

$$\begin{aligned} H &= (p_1^2 + p_2^2 + p_3^2) / 2 + p_1 q_2 - q_1 p_2 + \frac{q}{1+q} \cdot q_1 - q_1^2 - q_2^2 - \\ &\quad - \frac{1}{1+q} \cdot \frac{1}{r_1} - \frac{q}{1+q} \cdot \frac{1}{r_2} \end{aligned} \quad (2)$$

and explicitly the Hamiltonian equations are

$$\begin{aligned} dq_1 / dt &= \partial H / \partial p_1 = p_1 + q_2, \\ dq_2 / dt &= \partial H / \partial p_2 = p_2 - q_1, \\ dq_3 / dt &= \partial H / \partial p_3 = p_3, \\ dp_1 / dt &= -\partial H / \partial q_1 = p_2 + 2q_1 - \frac{q}{1+q} - \frac{1}{1+q} \cdot \frac{q_1}{r_1^3} - \frac{q}{1+q} \cdot \frac{(q_1-1)}{r_2^3}, \\ dp_2 / dt &= -\partial H / \partial q_2 = -p_1 + 2q_2 - \frac{1}{1+q} \cdot \frac{q_2}{r_1^3} - \frac{q}{1+q} \cdot \frac{q_2}{r_2^3}, \\ dp_3 / dt &= -\partial H / \partial q_3 = -\frac{1}{1+q} \cdot \frac{q_3}{r_1^3} - \frac{q}{1+q} \cdot \frac{q_3}{r_2^3}. \end{aligned} \quad (3)$$

CASE 2: In the $(q_{s1} O_2 q_{s2} q_{s3})$ coordinate system (with usual notation) the equations of motion are

$$\begin{aligned}
\frac{d^2 q_{s1}}{dt^2} + 2 \frac{dq_{s2}}{dt} &= q_{s1} - \frac{q'}{1+q'} - \frac{1}{1+q'} \cdot \frac{q_{s1}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{(q_{s1}-1)}{r_{s2}^3} \\
\frac{d^2 q_{s2}}{dt^2} + 2 \frac{dq_{s1}}{dt} &= q_{s2} - \frac{1}{1+q'} \cdot \frac{q_{s2}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{q_{s2}}{r_{s2}^3} \\
\frac{d^2 q_{s3}}{dt^2} &= -\frac{1}{1+q'} \cdot \frac{q_{s3}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{q_{s3}}{r_{s2}^3}
\end{aligned} \tag{4}$$

where $r_{s1} = \sqrt{q_{s1}^2 + q_{s2}^2 + q_{s3}^2}$, $r_{s2} = \sqrt{(q_{s1}-1)^2 + q_{s2}^2 + q_{s3}^2}$.

The Hamiltonian of "similar" (CRSTBP) system (4) is:

$$\begin{aligned}
H_s &= \frac{1}{2}(p_{s1}^2 + p_{s2}^2 + p_{s3}^2) - p_{s1}q_{s2} + q_{s1}p_{s2} + \frac{q'}{1+q'} q_{s1} - q_{s1}^2 - q_{s2}^2 - \\
&\quad - \frac{1}{1+q'} \cdot \frac{1}{r_{s1}} - \frac{q'}{1+q'} \cdot \frac{1}{r_{s2}}
\end{aligned} \tag{5}$$

and explicitly the "similar" canonical equations may be written

$$\begin{aligned}
\frac{dq_{s1}}{dt} &= \frac{\partial H_s}{\partial p_{s1}} = p_{s1} - q_{s2} \\
\frac{dq_{s2}}{dt} &= \frac{\partial H_s}{\partial p_{s2}} = p_{s2} + q_{s1} \\
\frac{dq_{s3}}{dt} &= \frac{\partial H_s}{\partial p_{s3}} = p_{s3} \\
\frac{dp_{s1}}{dt} &= -\frac{\partial H_s}{\partial q_{s1}} = -p_{s2} + 2q_{s1} - \frac{q'}{1+q'} - \frac{1}{1+q'} \cdot \frac{q_{s1}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{(q_{s1}-1)}{r_{s2}^3} \\
\frac{dp_{s2}}{dt} &= -\frac{\partial H_s}{\partial q_{s2}} = p_{s1} + 2q_{s2} - \frac{1}{1+q'} \cdot \frac{q_{s2}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{q_{s2}}{r_{s2}^3} \\
\frac{dp_{s3}}{dt} &= -\frac{\partial H_s}{\partial q_{s3}} = -\frac{1}{1+q'} \cdot \frac{q_{s3}}{r_{s1}^3} - \frac{q'}{1+q'} \cdot \frac{q_{s3}}{r_{s2}^3}
\end{aligned} \tag{6}$$

2. COORDINATE TRANSFORMATION (CASE 1)

Several regularizing transformations are known. For the regularization of the equations of motion (1) we introduce the generating function S in the frame (q_1, O_1, q_2, q_3)

in the following form (Stiefel and Scheifele 1971)

$$S = p_1 f(Q_1, Q_2) + p_2 g(Q_1, Q_2), \quad (7)$$

where f and g are harmonic conjugated functions, with the property

$$\begin{aligned} \frac{\partial f}{\partial Q_1} &= \frac{\partial g}{\partial Q_2} \\ \frac{\partial f}{\partial Q_2} &= -\frac{\partial g}{\partial Q_1} \end{aligned} \quad (8)$$

The generating equations are

$$\begin{aligned} q_i &= \frac{\partial S}{\partial p_i}, \quad i = 1, 2 \\ P_i &= \frac{\partial S}{\partial Q_i} \end{aligned} \quad (9)$$

or explicitly

$$\begin{aligned} q_1 &= \frac{\partial S}{\partial p_1} = f(Q_1, Q_2) \\ q_2 &= \frac{\partial S}{\partial p_2} = g(Q_1, Q_2) \\ P_1 &= \frac{\partial S}{\partial Q_1} = p_1 \frac{\partial f}{\partial Q_1} + p_2 \frac{\partial g}{\partial Q_1} = p_1 a_{11} + p_2 a_{12} \\ P_2 &= \frac{\partial S}{\partial Q_2} = p_1 \frac{\partial f}{\partial Q_2} + p_2 \frac{\partial g}{\partial Q_2} = -p_1 a_{12} + p_2 a_{11} \end{aligned} \quad (10)$$

where

$$\begin{aligned} a_{11} &= \frac{\partial f}{\partial Q_1} = \frac{\partial g}{\partial Q_2} \\ a_{12} &= -\frac{\partial f}{\partial Q_2} = \frac{\partial g}{\partial Q_1} \end{aligned} \quad (11)$$

Let introduce the following notation, (Szebehely 1967)

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ -a_{12} & a_{11} \end{pmatrix}, \quad D = \det \mathbf{A} = a_{11}^2 + a_{12}^2,$$

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad \mathbf{p}^2 = \frac{1}{D} \mathbf{P}^2 \quad (12)$$

The new Hamiltonian of the (CRTBP) system of equations (1) is

$$\begin{aligned} \bar{H} = & \frac{1}{2D} \left[P_1^2 + P_2^2 + P_1 \frac{\partial}{\partial Q_2} (f^2 + g^2) - P_2 \frac{\partial}{\partial Q_1} (f^2 + g^2) \right] + \frac{q}{1+q} f - \\ & - f^2 - g^2 - \frac{1}{1+q} \frac{1}{\bar{r}_1} - \frac{q}{1+q} \frac{1}{\bar{r}_2} \end{aligned} \quad (13)$$

where

$$\bar{r}_1 = \sqrt{f^2 + g^2}, \quad \bar{r}_2 = \sqrt{(f-1)^2 + g^2}, \quad D = 4(Q_1^2 + Q_2^2)$$

and the Hamiltonian equations in new variables become

$$\begin{aligned} \frac{dQ_1}{dt} &= \frac{1}{2D} \left[2P_1 + \frac{\partial}{\partial Q_2} (f^2 + g^2) \right] \\ \frac{dQ_2}{dt} &= \frac{1}{2D} \left[2P_2 - \frac{\partial}{\partial Q_1} (f^2 + g^2) \right] \\ \frac{dP_1}{dt} &= -\frac{\partial \bar{H}}{\partial Q_1} = -\frac{P_1}{2D} \frac{\partial}{\partial Q_1 \partial Q_2} (f^2 + g^2) + \frac{P_2}{2D} \frac{\partial}{\partial Q_1 \partial Q_1} (f^2 + g^2) - \frac{q}{1+q} \frac{\partial f}{\partial Q_1} + \\ &+ \frac{\partial}{\partial Q_1} (f^2 + g^2) + \frac{1}{1+q} \frac{\partial}{\partial Q_1} \left(\frac{1}{\bar{r}_1} \right) + \frac{q}{1+q} \frac{\partial}{\partial Q_1} \left(\frac{1}{\bar{r}_2} \right) \\ \frac{dP_2}{dt} &= -\frac{\partial \bar{H}}{\partial Q_2} = -\frac{P_1}{2D} \frac{\partial}{\partial Q_2 \partial Q_2} (f^2 + g^2) + \frac{P_2}{2D} \frac{\partial}{\partial Q_2 \partial Q_1} (f^2 + g^2) - \frac{q}{1+q} \frac{\partial f}{\partial Q_2} + \\ &+ \frac{\partial}{\partial Q_2} (f^2 + g^2) + \frac{1}{1+q} \frac{\partial}{\partial Q_2} \left(\frac{1}{\bar{r}_1} \right) + \frac{q}{1+q} \frac{\partial}{\partial Q_2} \left(\frac{1}{\bar{r}_2} \right) \end{aligned} \quad (14)$$

Because the singularity of the problem is given by the terms $1/\bar{r}_1$ and $1/\bar{r}_2$, we will make a global regularization, (Castilho and Vidal 1999; Csillik 2003), in the $P_1(m_1,0)$ and $P_2(m_2,0)$ points using Levi-Civita's transformation

$$f = Q_1^2 - Q_2^2, \quad g = 2Q_1Q_2 \quad (15)$$

Using Levi-Civita's transformation, (Mioc and Csillik 2002), the new Hamiltonian equations are

$$\begin{aligned} \frac{dQ_1}{dt} &= \frac{P_1}{D} + \frac{Q_2}{2} \\ \frac{dQ_2}{dt} &= \frac{P_2}{D} - \frac{Q_1}{2} \\ \frac{dP_1}{dt} &= \frac{P_2}{2} + \frac{Q_1(P_1^2 + P_2^2)}{4\bar{r}_1^2} - \frac{2qQ_1}{1+q} + Q_1D - \frac{2}{(1+q)} \frac{Q_1}{\bar{r}_1^2} - \frac{2q}{(1+q)} \frac{Q_1(\bar{r}_1 - 1)}{\bar{r}_2^3} \\ \frac{dP_2}{dt} &= -\frac{P_1}{2} + \frac{Q_2(P_1^2 + P_2^2)}{4\bar{r}_1^2} + \frac{2qQ_2}{1+q} + Q_2D - \frac{2}{(1+q)} \frac{Q_2}{\bar{r}_1^2} - \frac{2q}{(1+q)} \frac{Q_2(\bar{r}_1 + 1)}{\bar{r}_2^3} \end{aligned} \quad (16)$$

where

$$\bar{r}_1 = |Q_1^2 + Q_2^2|, \quad \bar{r}_2 = \sqrt{(Q_1^2 - Q_2^2 - 1)^2 + 4Q_1^2Q_2^2} \quad (17)$$

with the new Hamiltonian

$$\begin{aligned} \bar{H} &= \frac{1}{8(Q_1^2 + Q_2^2)} [P_1^2 + P_2^2] + \frac{1}{2} [P_1Q_2 - P_2Q_1] + \frac{q}{1+q} (Q_1^2 - Q_2^2) - Q_1^4 - Q_2^4 \\ &\quad - 2Q_1^2Q_2^2 - \frac{1}{1+q} \cdot \frac{1}{Q_1^2 + Q_2^2} - \frac{q}{1+q} \cdot \frac{1}{\sqrt{(Q_1^2 - Q_2^2 - 1)^2 + 4Q_1^2Q_2^2}} \end{aligned} \quad (18)$$

3. TIME TRANSFORMATION (CASE 1)

To solve the above equations, we introduce the fictitious time τ (Waldvogel 1982; Érdi 2004), and making the time transformation

$$\frac{dt}{d\tau} = \bar{r}_1^{-2} \bar{r}_2^{-3}$$

the new regular equations of motion are

$$\begin{aligned}
\frac{dQ_1}{d\tau} &= \frac{P_1 \bar{r}_1 \bar{r}_2^3}{4} + \frac{Q_2 \bar{r}_1^2 \bar{r}_2^3}{2} \\
\frac{dQ_2}{d\tau} &= \frac{P_2 \bar{r}_1 \bar{r}_2^3}{4} - \frac{Q_1 \bar{r}_1^2 \bar{r}_2^3}{2} \\
\frac{dP_1}{d\tau} &= \frac{P_2 \bar{r}_1^2 \bar{r}_2^3}{2} + \frac{Q_1 (P_1^2 + P_2^2) \bar{r}_2^3}{4} - \frac{2qQ_1}{1+q} \bar{r}_1^2 \bar{r}_2^3 + 4Q_1 \bar{r}_1^3 \bar{r}_2^3 - \\
&\quad - \frac{2Q_1 \bar{r}_2^3}{(1+q)} - \frac{2qQ_1 (\bar{r}_1 - 1) \bar{r}_1^2}{(1+q)} \\
\frac{dP_2}{d\tau} &= -\frac{P_1 \bar{r}_1^2 \bar{r}_2^3}{2} + \frac{Q_2 (P_1^2 + P_2^2) \bar{r}_2^3}{4} + \frac{2qQ_2}{1+q} \bar{r}_1^2 \bar{r}_2^3 + 4Q_2 \bar{r}_1^3 \bar{r}_2^3 - \\
&\quad - \frac{2Q_2 \bar{r}_2^3}{(1+q)} - \frac{2qQ_2 (\bar{r}_1 + 1) \bar{r}_1^2}{(1+q)}
\end{aligned} \tag{19}$$

The explicit equations of motion for RCRTBP may be written

$$\begin{aligned}
\frac{d^2 Q_1}{d\tau^2} &= \frac{1}{4} \frac{dP_1}{d\tau} \bar{r}_1 \bar{r}_2^3 + \frac{1}{2} \frac{dQ_2}{d\tau} \bar{r}_1^2 \bar{r}_2^3 + \left(\frac{P_1}{2} + 2Q_2 \bar{r}_1 \right) \left(Q_1 \frac{dQ_1}{d\tau} + Q_2 \frac{dQ_2}{d\tau} \right) \bar{r}_2^3 + \\
&\quad + 3 \left(\frac{P_1}{2} + Q_2 \bar{r}_1 \right) \left((Q_1^3 + Q_1 Q_2^2 - Q_1) \frac{dQ_1}{d\tau} + (Q_2^3 + Q_1^2 Q_2 + Q_2) \frac{dQ_2}{d\tau} \right) \bar{r}_1 \bar{r}_2 \\
\frac{d^2 Q_2}{d\tau^2} &= \frac{1}{4} \frac{dP_2}{d\tau} \bar{r}_1 \bar{r}_2^3 - \frac{1}{2} \frac{dQ_1}{d\tau} \bar{r}_1^2 \bar{r}_2^3 + \left(\frac{P_2}{2} - 2Q_1 \bar{r}_1 \right) \left(Q_1 \frac{dQ_1}{d\tau} + Q_2 \frac{dQ_2}{d\tau} \right) \bar{r}_2^3 + \\
&\quad + 3 \left(\frac{P_2}{2} - Q_1 \bar{r}_1 \right) \left((Q_1^3 + Q_1 Q_2^2 - Q_1) \frac{dQ_1}{d\tau} + (Q_2^3 + Q_1^2 Q_2 + Q_2) \frac{dQ_2}{d\tau} \right) \bar{r}_1 \bar{r}_2
\end{aligned} \tag{20}$$

For the application of the above problem in a binary system, we can obtain the solution in the form

$$\begin{aligned}
q_1(t) &= Q_1^2(t) - Q_2^2(t) \\
q_2(t) &= 2Q_1(t)Q_2(t)
\end{aligned} \tag{21}$$

4. COORDINATE TRANSFORMATION (CASE 2)

For the coordinate transformation in the second case we introduce the generating function S_s in the plane $q_{s1}Oq_{s2}$ in the following form

$$S_s = p_{s1} f_s(Q_{s1}, Q_{s2}) + p_{s2} g_s(Q_{s1}, Q_{s2}) \quad (22)$$

where f_s and g_s are harmonic conjugated functions, with the property

$$\begin{aligned} \frac{\partial f_s}{\partial Q_{s1}} &= \frac{\partial g_s}{\partial Q_{s2}} \\ \frac{\partial f_s}{\partial Q_{s2}} &= -\frac{\partial g_s}{\partial Q_{s1}} \end{aligned} \quad (23)$$

The generating equations are

$$\begin{aligned} q_{si} &= \frac{\partial S_s}{\partial p_{si}}, \quad i = 1, 2 \\ P_{si} &= \frac{\partial S_s}{\partial Q_{si}} \end{aligned} \quad (24)$$

or explicitly

$$\begin{aligned} q_{s1} &= \frac{\partial S_s}{\partial p_{s1}} = f_s(Q_{s1}, Q_{s2}) \\ q_{s2} &= \frac{\partial S_s}{\partial p_{s2}} = g_s(Q_{s1}, Q_{s2}) \\ P_{s1} &= \frac{\partial S_s}{\partial Q_{s1}} = p_{s1} \frac{\partial f_s}{\partial Q_{s1}} + p_{s2} \frac{\partial g_s}{\partial Q_{s1}} = p_{s1} b_{11} + p_{s2} b_{12} \\ P_{s2} &= \frac{\partial S_s}{\partial Q_{s2}} = p_{s1} \frac{\partial f_s}{\partial Q_{s2}} + p_{s2} \frac{\partial g_s}{\partial Q_{s2}} = -p_{s1} b_{12} + p_{s2} b_{11} \end{aligned} \quad (25)$$

where

$$\begin{aligned} b_{11} &= \frac{\partial f_s}{\partial Q_{s1}} = \frac{\partial g_s}{\partial Q_{s2}} \\ b_{12} &= -\frac{\partial f_s}{\partial Q_{s2}} = \frac{\partial g_s}{\partial Q_{s1}} \end{aligned} \quad (26)$$

Let introduce the following notation

$$\begin{aligned}
B &= \begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{11} \end{pmatrix}, \quad D_s = \det \mathbf{B} = b_{11}^2 + b_{12}^2, \\
\mathbf{p}_s &= \begin{pmatrix} p_{s1} \\ p_{s2} \end{pmatrix}, \quad \mathbf{P}_s = \begin{pmatrix} P_{s1} \\ P_{s2} \end{pmatrix}, \quad \mathbf{p}_s^2 = \frac{1}{D_s} \mathbf{P}_s^2
\end{aligned} \tag{27}$$

The new Hamiltonian for the case 2 may be written

$$\begin{aligned}
\bar{H}_s &= \frac{1}{2D_s} \left[P_{s1}^2 + P_{s2}^2 - P_{s1} \frac{\partial}{\partial Q_{s2}} (f_s^2 + g_s^2) + P_{s2} \frac{\partial}{\partial Q_{s1}} (f_s^2 + g_s^2) \right] + \\
&+ \frac{q'}{1+q'} f_s - f_s^2 - g_s^2 - \frac{1}{1+q'} \frac{1}{\bar{r}_{s1}} - \frac{q'}{1+q'} \frac{1}{\bar{r}_{s2}}
\end{aligned} \tag{28}$$

where

$$\bar{r}_{s1} = \sqrt{f_s^2 + g_s^2}, \quad \bar{r}_{s2} = \sqrt{(f_s - 1)^2 + g_s^2}$$

and the canonical equations in new variables become

$$\begin{aligned}
\frac{dQ_{s1}}{dt} &= \frac{1}{2D_s} \left[2P_{s1} - \frac{\partial}{\partial Q_{s2}} (f_s^2 + g_s^2) \right] \\
\frac{dQ_{s2}}{dt} &= \frac{1}{2D_s} \left[2P_{s2} + \frac{\partial}{\partial Q_{s1}} (f_s^2 + g_s^2) \right] \\
\frac{dP_{s1}}{dt} &= -\frac{\partial \bar{H}_s}{\partial Q_{s1}} = \frac{P_{s1}}{2D} \frac{\partial}{\partial Q_{s1} \partial Q_{s2}} (f_s^2 + g_s^2) - \frac{P_{s2}}{2D} \frac{\partial}{\partial Q_{s1} \partial Q_{s1}} (f_s^2 + g_s^2) - \\
&- \frac{q'}{1+q'} \frac{\partial f_s}{\partial Q_{s1}} + \frac{\partial}{\partial Q_{s1}} (f_s^2 + g_s^2) + \frac{1}{1+q'} \frac{\partial}{\partial Q_{s1}} \left(\frac{1}{\bar{r}_{s1}} \right) + \frac{q'}{1+q'} \frac{\partial}{\partial Q_{s1}} \left(\frac{1}{\bar{r}_{s2}} \right) \\
\frac{dP_{s2}}{dt} &= -\frac{\partial \bar{H}_s}{\partial Q_{s2}} = \frac{P_{s1}}{2D} \frac{\partial}{\partial Q_{s2} \partial Q_{s2}} (f_s^2 + g_s^2) - \frac{P_{s2}}{2D} \frac{\partial}{\partial Q_{s2} \partial Q_{s1}} (f_s^2 + g_s^2) - \\
&- \frac{q'}{1+q'} \frac{\partial f_s}{\partial Q_{s2}} + \frac{\partial}{\partial Q_{s2}} (f_s^2 + g_s^2) + \frac{1}{1+q'} \frac{\partial}{\partial Q_{s2}} \left(\frac{1}{\bar{r}_{s1}} \right) + \frac{q'}{1+q'} \frac{\partial}{\partial Q_{s2}} \left(\frac{1}{\bar{r}_{s2}} \right)
\end{aligned} \tag{29}$$

Because the singularity of the problem is given by $1/\bar{r}_{s1}$ and $1/\bar{r}_{s2}$, we made a global regularization using Levi-Civita's transformation

$$\begin{aligned} f_s &= Q_{s1}^2 - Q_{s2}^2 \\ g_s &= 2Q_{s1}Q_{s2} \end{aligned} \quad (30)$$

The similar Hamiltonian equations are given by

$$\begin{aligned} \frac{dQ_{s1}}{dt} &= \frac{P_{s1}}{D_s} - \frac{Q_{s2}}{2} \\ \frac{dQ_{s2}}{dt} &= \frac{P_{s2}}{D_s} + \frac{Q_{s1}}{2} \\ \frac{dP_{s1}}{dt} &= -\frac{P_{s2}}{2} + \frac{Q_{s1}(P_{s1}^2 + P_{s2}^2)}{4r_{s1}^2} - \frac{2q'Q_{s1}}{1+q'} + Q_{s1}D_s - \\ &\quad - \frac{2}{(1+q')} \frac{Q_{s1}}{r_{s1}^2} - \frac{2q'}{(1+q')} \frac{Q_{s1}(r_{s1}-1)}{r_{s2}^3} \\ \frac{dP_{s2}}{dt} &= \frac{P_{s1}}{2} + \frac{Q_{s2}(P_{s1}^2 + P_{s2}^2)}{4r_{s1}^2} + \frac{2q'Q_{s2}}{1+q'} + Q_{s2}D_s - \\ &\quad - \frac{2}{(1+q')} \frac{Q_{s2}}{r_{s1}^2} - \frac{2q'}{(1+q')} \frac{Q_{s2}(r_{s1}+1)}{r_{s2}^3} \end{aligned} \quad (31)$$

with the new Hamiltonian

$$\begin{aligned} \bar{H}_s &= \frac{P_{s1}^2 + P_{s2}^2}{8(Q_{s1}^2 + Q_{s2}^2)} + \frac{-P_{s1}Q_{s2} + P_{s2}Q_{s1}}{2} + \frac{q'}{1+q'}(Q_{s1}^2 - Q_{s2}^2) - Q_{s1}^4 - Q_{s2}^4 - 2Q_{s1}^2Q_{s2}^2 \\ &\quad - \frac{1}{(1+q')} \frac{1}{Q_{s1}^2 + Q_{s2}^2} - \frac{q'}{(1+q')} \frac{1}{\sqrt{(Q_{s1}^2 - Q_{s2}^2 - 1)^2 + 4Q_{s1}^2Q_{s2}^2}} \end{aligned} \quad (32)$$

5. TIME TRANSFORMATION (CASE 2)

Introducing the fictitious time τ and making the time transformation the new regular equations of motion are obtained in the form

$$\frac{dt}{d\tau} = r_{s1}^2 r_{s2}^3$$

the new regular equations of motion are obtained in the form

$$\begin{aligned}
\frac{dQ_{s1}}{d\tau} &= \frac{P_{s1}\bar{r}_{s1}\bar{r}_{s2}^3}{4} - \frac{Q_{s2}\bar{r}_{s1}^2\bar{r}_{s2}^3}{2} \\
\frac{dQ_{s2}}{d\tau} &= \frac{P_{s2}\bar{r}_{s1}\bar{r}_{s2}^3}{4} + \frac{Q_{s1}\bar{r}_{s1}^2\bar{r}_{s2}^3}{2} \\
\frac{dP_{s1}}{d\tau} &= -\frac{P_{s2}\bar{r}_{s1}^2\bar{r}_{s2}^3}{2} + \frac{Q_{s1}(P_{s1}^2 + P_{s2}^2)\bar{r}_{s2}^3}{4} - \frac{2q_s Q_{s1}\bar{r}_{s1}^2\bar{r}_{s2}^3}{1+q_s} + 4Q_{s1}\bar{r}_{s1}^3\bar{r}_{s2}^3 - \\
&\quad - \frac{2Q_{s1}\bar{r}_{s2}^3}{(1+q_s)} - \frac{2q_s Q_{s1}(\bar{r}_{s1}-1)\bar{r}_{s1}^2}{(1+q_s)} \\
\frac{dP_{s2}}{d\tau} &= \frac{P_{s1}\bar{r}_{s1}^2\bar{r}_{s2}^3}{2} + \frac{Q_{s2}(P_{s1}^2 + P_{s2}^2)\bar{r}_{s2}^3}{4} + \frac{2q_s Q_{s2}\bar{r}_{s1}^2\bar{r}_{s2}^3}{1+q_s} + 4Q_{s2}\bar{r}_{s1}^3\bar{r}_{s2}^3 - \\
&\quad - \frac{2Q_{s2}\bar{r}_{s2}^3}{(1+q_s)} - \frac{2q_s Q_{s2}(\bar{r}_{s1}+1)\bar{r}_{s1}^2}{(1+q_s)}
\end{aligned} \tag{33}$$

$$\bar{r}_{s1} = |Q_{s1}^2 + Q_{s2}^2|, \quad \bar{r}_{s2} = \sqrt{(Q_{s1}^2 - Q_{s2}^2 - 1)^2 + 4Q_{s1}^2 Q_{s2}^2}$$

The explicit equations of motion for RCRSTBP are given by

$$\begin{aligned}
\frac{d^2 Q_{s1}}{d\tau^2} &= \frac{1}{4} \frac{dP_{s1}}{d\tau} \bar{r}_{s1} \bar{r}_{s2}^3 - \frac{1}{2} \frac{dQ_{s2}}{d\tau} \bar{r}_{s1}^2 \bar{r}_{s2}^3 + \left(\frac{P_{s1}}{2} - 2Q_{s2} \bar{r}_{s1} \right) \left(Q_{s1} \frac{dQ_{s1}}{d\tau} + Q_{s2} \frac{dQ_{s2}}{d\tau} \right) \bar{r}_{s2}^3 + \\
&+ 3 \left(\frac{P_{s1}}{2} - Q_{s2} \bar{r}_{s1} \right) \left((Q_{s1}^3 + Q_{s1} Q_{s2}^2 - Q_{s1}) \frac{dQ_{s1}}{d\tau} + (Q_{s2}^3 + Q_{s1}^2 Q_{s2} + Q_{s2}) \frac{dQ_{s2}}{d\tau} \right) \bar{r}_{s1} \bar{r}_{s2} \\
\frac{d^2 Q_{s2}}{d\tau^2} &= \frac{1}{4} \frac{dP_{s2}}{d\tau} \bar{r}_{s1} \bar{r}_{s2}^3 + \frac{1}{2} \frac{dQ_{s1}}{d\tau} \bar{r}_{s1}^2 \bar{r}_{s2}^3 + \left(\frac{P_{s2}}{2} + 2Q_{s1} \bar{r}_{s1} \right) \left(Q_{s1} \frac{dQ_{s1}}{d\tau} + Q_{s2} \frac{dQ_{s2}}{d\tau} \right) \bar{r}_{s2}^3 + \\
&+ 3 \left(\frac{P_{s2}}{2} + Q_{s1} \bar{r}_{s1} \right) \left((Q_{s1}^3 + Q_{s1} Q_{s2}^2 - Q_{s1}) \frac{dQ_{s1}}{d\tau} + (Q_{s2}^3 + Q_{s1}^2 Q_{s2} + Q_{s2}) \frac{dQ_{s2}}{d\tau} \right) \bar{r}_{s1} \bar{r}_{s2}
\end{aligned} \tag{34}$$

For the application of the above "similar" problem in a binary system, we can obtain the solution in the form

$$\begin{aligned}
q_{s1}(t) &= Q_{s1}^2(t) - Q_{s2}^2(t) \\
q_{s2}(t) &= 2Q_{s1}(t)Q_{s2}(t)
\end{aligned} \tag{35}$$

6. NUMERICAL EXPERIMENTS

For the numerical integration (Earth-Moon binary system) we use the initial values

$$q_{10} = -0.4, \quad q_{20} = 0.4, \quad p_{10} = 0, \quad p_{20} = -0.5, \quad t \in [0, 2\pi], \quad q = 0.0123$$

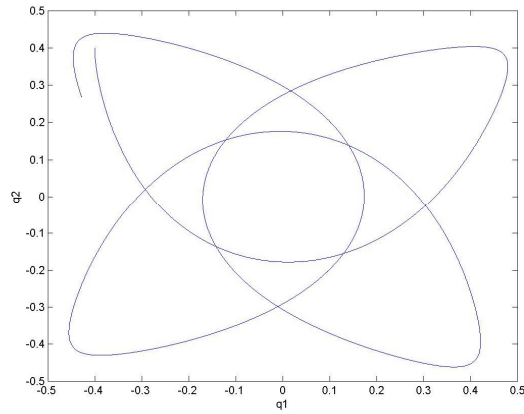


Fig. 1 – Illustration of the motion in CRTBP.

For the numerical integration (Earth-Moon binary system) in the “similar” coordinate system we use the initial values

$$q_{s10} = 0.6, \quad q_{s20} = 0.4, \quad p_{s10} = 0, \quad p_{s20} = 0.5, \quad t \in [0, 2\pi], \quad q' = 81.45$$

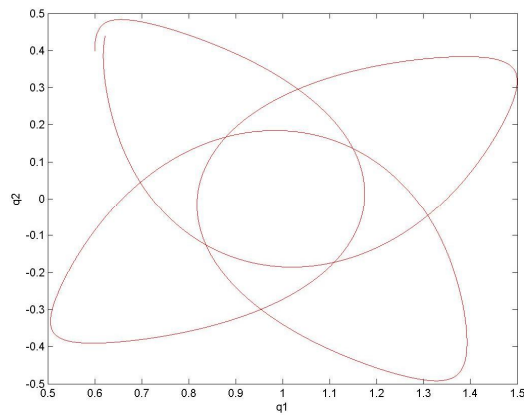


Fig. 2 – Illustration of the motion in CRSTBP.

For the numerical integration (Earth-Moon binary system) in the regularized coordinate system we use the initial values in case RCRTBP

$$Q_{s10} = \pm 0.2878240999, \quad Q_{s20} = \pm 0.6948688455, \\ P_{s10} = 0, \quad P_{s20} = \pm 0.7521206187, \quad \tau \in [0, 2\pi]$$

and in the case RCRSTBP

$$Q_{s10} = \pm 0.8127454260, \quad Q_{s20} = \pm 0.2460795147, \\ P_{s10} = 0, \quad P_{s20} = \pm 0.8491821094, \quad \tau \in [0, 2\pi]$$

7. CONCLUDING REMARKS

Many papers in the last decade have studied the restricted three-body system in a phase space. During these studies, difficulties have arisen when the system approaches a close encounter.

Using the regularization method in the similar coordinates system we gave explicitly the equations of motion for RCRTBP and RCRSTBP. We studied numerically the regular equations of motion, we wrote them in canonical form, and we obtained that the integrator using regularized equations of motion is more efficient.

Our method may provide new directions for studies of circular restricted three-body problem integration using similar coordinate systems. It is an important tool for developing efficient numerical algorithms.

APPENDIX: MATLAB CODE

```
function dy=crtbp(t,y)      %Save as crtbp.m
r1=sqrt(y(1)^2+y(3)^2);
r2=sqrt((1-y(1))^2+y(3)^2);
q=0.0123;
q1=q/(1+q);
q2=1/(1+q);
dy(1)=y(2);
dy(2)=2*y(4)+y(1)-q1-q2*y(1)/(r1)^3-q1*(y(1)-1)/(r2)^3;
dy(3)=y(4);
dy(4)=-2*y(2)+y(3)-q2*y(3)/(r1)^3-q1*y(3)/(r2)^3;
dy=dy';
function output=crtbp;      %Save as crtbp1.m; Call: crtbp1
```

```

y0=[-0.4, 0, 0.4, -0.5];
tspan = [0:0.0001:2*pi];
[t,y]=ode23('crstbp',tspan,y0);

```

```

function dy=crstbp(t,y)      %Save as crstbp.m
r1=sqrt(y(1)^2+y(3)^2);
r2=sqrt((1-y(1))^2+y(3)^2);
qs=81.45;
q1=qs/(1+qs);
q2=1/(1+qs);
dy(1)=y(2);
dy(2)=-2*y(4)+y(1)-q1-q2*y(1)/(r1)^3-q1*(y(1)-1)/(r2)^3;
dy(3)=y(4);
dy(4)=2*y(2)+y(3)-q2*y(3)/(r1)^3-q1*y(3)/(r2)^3;
dy=dy';
function output=crstbp;      %Save as crstbp1.m; Call: crstbp1
y0=[0.6, 0, 0.4, 0.5];
tspan = [0:0.0001:2*pi];
[t,y]=ode45('crstbp',tspan,y0);

```

```

function dx=regucrtbp(t,x)    % Save as regucrtbp.m
q=0.0123;
r1=x(1)*x(1)+x(3)*x(3);
r2=sqrt((x(1)*x(1)-x(3)*x(3)-1)^2+4*x(1)*x(1)*x(3)*x(3));
dx(1)=((x(2)+2*x(3)*r1)*r1*r2^3)/4;
dx(3)=((x(4)-2*x(1)*r1)*r1*r2^3)/4;
dx(2)=((((x(4)*r1^2*r2^3)/2+x(1)*(x(2)^2+x(4)^2)*r2^3/4-
2*q*r1^2*r2^3*x(1)/(1+q)+4*r1^3*r2^3*x(1)-2*x(1)*r2^3/(1+q)-
2*q*x(1)*r1^2*(r1-1)/(1+q))*r1*r2^3)/4+r1^2*(r2^3)*dx(3)/2+(x(2)/2+2*x(3)*r1)*
(x(1)*dx(1)+x(3)*dx(3))*r2^3+3*r1*r2*(x(2)/2+x(3)*r1)*((x(1)^3-x(1)+
x(1)*x(3)^2)*dx(1)+(x(3)^3+x(3)+x(1)^2*x(3))*dx(3));
dx(4)=((((-x(2)*r1^2*r2^3)/2+x(3)*(x(2)^2+x(4)^2)*r2^3/4+
2*q*r1^2*r2^3*x(3)/(1+q)+4*r1^3*r2^3*x(3)-2*x(3)*(r2^3)/(1+q)-
2*q*x(3)*r1^2*(r1+1)/(1+q))*r1*r2^3)/4-r1^2*r2^3*dx(1)/2+(x(4)/2-2*x(1)*r1)*
(x(1)*dx(1)+x(3)*dx(3))*r2^3+3*r1*r2*(x(4)/2-x(1)*r1)*((x(1)^3-x(1)+
x(1)*x(3)^2)*dx(1)+(x(3)^3+x(3)+x(1)^2*x(3))*dx(3));
dx=dx';
function output=regucrtbp;    %Save as regucrtbp1.m; Call: regucrtbp1
x0=[0.2878240999, 0, 0.6948688455, 0.7521206187]';
tspan = [0:0.001:2*pi];

```

```
[t,x]=ode45('regucrtbp',tspan,x0);
```

```
function dx=regucrstbp(t,x)      % Save as regucrstbp.m
q=81.45;
r1=x(1)^2+x(3)^2;
r2=(sqrt((x(1)*x(1)-x(3)*x(3)-1)^2+4*x(1)*x(1)*x(3)*x(3)));
dx(1)=((x(2)-2*x(3)*r1)*r1*r2^3)/4;
dx(3)=((x(4)+2*x(1)*r1)*r1*r2^3)/4;
dx(2)=((((-x(4)*r1^2*r2^3)/2+x(1)*(x(2)^2+x(4)^2)*r2^3/4-
2*q*r1^2*r2^3*x(1)/(1+q)+4*r1^3*r2^3*x(1)-2*x(1)*r2^3/(1+q)-
2*q*x(1)*r1^2*(r1-1)/(1+q))*r1*r2^3)/4-r1^2*(r2^3)*dx(3)/2+(x(2)/2-2*x(3)*r1)*
(x(1)*dx(1)+x(3)*dx(3))*r2^3+3*r1*r2*(x(2)/2-x(3)*r1)*((x(1)^3-x(1)+
x(1)*x(3)^2)*dx(1)+(x(3)^3+x(3)+x(1)^2*x(3))*dx(3));
dx(4)=((((x(2)*r1^2*r2^3)/2+x(3)*(x(2)^2+x(4)^2)*r2^3/4+
2*q*r1^2*r2^3*x(3)/(1+q)+4*r1^3*r2^3*x(3)-2*x(3)*(r2^3)/(1+q)-
2*q*x(3)*r1^2*(r1+1)/(1+q))*r1*r2^3)/4+r1^2*r2^3*dx(1)/2+(x(4)/2+2*x(1)*r1)*
(x(1)*dx(1)+x(3)*dx(3))*r2^3+3*r1*r2*(x(4)/2+x(1)*r1)*((x(1)^3-x(1)+
x(1)*x(3)^2)*dx(1)+(x(3)^3+x(3)+x(1)^2*x(3))*dx(3));
dx=dx';
function output=regucrstbp;      %Save as regucrstbp1.m; Call: regucrstbp1
x0=[0.8127454260, 0, 0.2460795147, 0.8491821094]';
tspan = [0:0.001:2*pi];
[t,x]=ode45('regucrstbp',tspan,x0);
```

REFERENCES

- Castilho, C, Vidal, C.: 1999, *Qualitative Theory Dyn. Syst.*, **1**, 1.
- Celletti, A. et al.: 2011, *Celest.Mech. Dyn. Astron.*, **109**, 265.
- Csillik, I.: 2003, *Regularization Methods in Celestial Mechanics*, Casa Cărții de Știință Press, Cluj (in Romanian).
- Érdi, B.: 2004, *Celest.Mech. Dyn. Astron.*, **90**, 35.
- Kopal, Z.: 1978, *Dynamics of Close Binary Systems*, D. Reidel Publ. Co., Dordrecht.
- Roman, R.: 2010, Communication presented at the Symposium "New Challenges in Astronomy and Astrophysics", Cluj-Napoca.
- Mioc, V., Csillik, I.: 2002, *Rom. Astron. J.*, **12**, 167.
- Stiefel, L., Scheifele, G.: 1971, *Linear and Regular Celestial Mechanics*, Springer Verlag, Berlin.
- Szebehely, V.: 1967, *Theory of Orbits*, Academic Press, New York.
- Waldvogel, J.: 1982, *Celest.Mech. Dyn. Astron.*, **28**, 69.
- Waldvogel, J.: 2006, *Celest.Mech. Dyn. Astron.*, **95**, 201.

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