SIMULTANEOUS TOTAL COLLAPSE IN FOCK'S FIELD

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Abstract. We tackle another aspect of particle dynamics in relativistic fields: the simultaneous total collision in the *n*-body problem attached to Fock's potential. Based on an equality concerning the moment of inertia and on the Lagrange–Jacobi relation transposed for this model, the impossibility of the simultaneous total collapse in infinite time is proved.

Key words: celestial mechanics – *n*-body problem – relativistic fields – collision.

1. INTRODUCTION

Fock's field (see Fock 1959) is one of the best known solutions of Einstein's field equations for the spherically symmetric case. The respective metric can be transposed into the realm of classical mechanics, via a standard canonical formalism (e.g., Landau and Lifshitz 1948; Chandrasekhar 1983). This leads to a Binet-type equation that provides the expression of the force of interaction between two particles in Fock's field. Since this force is conservative, it can be considered as deriving from a potential function that reads

$$U(\mathbf{r}) = \sum_{k=1}^{4} \frac{A_k}{|\mathbf{r}|^k}, \tag{1}$$

in which the expressions of the positive parameters A_k, $k = \overline{1, 4}$, are well established (see, e.g., Mioc 1994; Mioc and Pérez-Chavela 2008), while $\mathbf{r}$ denotes the radius vector of one particle with respect to another in this field.

In a previous paper (Mioc et al. 2011), we approached the *n*-body problem associated to a much more general model, which includes Fock's potential as a particular case. Considering it as a problem of classical celestial mechanics, we transposed some well-known results in this field: the Lagrange–Jacobi relation and Sundman's inequalities. Here we shall establish another important result, analogous to a classical one: the impossibility of simultaneous total collapse in infinite time.

2. BASIC EQUATIONS AND AUXILIARY RESULTS

In the *n*-body problem, the potential (1) can be specified as

$$U(\mathbf{r}) = \sum_{k=1}^{4} U_k(\mathbf{r}), \tag{2}$$

with

$$U_k(\mathbf{r}) = \sum_{1\le i<j\le n} \frac{A_{k,ij}}{r_{ij}^{\gamma_k}}. \tag{3}$$

Here $\gamma_k = k$, $k = \overline{1,4}$, $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n)$ is the configuration of the system, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the mutual distance between the *i*th and *j*thparticles, while $A_{k,ij}$ are positive parameters, functions of the masses m_i, m_j, and symmetrical with respect to these ones (hence to the indices *i* and *j*).

From now on, we denote simply $U(\mathbf{r}) = U(\mathbf{r}(t))$ and $U_k(\mathbf{r}) = U_k(\mathbf{r}(t))$ by U and U_k, respectively.

The equations of motion of the *n*-body system read

$$m_i\ddot{\mathbf{r}}_i = \frac{\partial U}{\partial \mathbf{r}_i} = -\sum_{1\le i<j\le n} (\mathbf{r}_i - \mathbf{r}_j) \sum_{k=1}^{4} \gamma_k \frac{A_{k,ij}}{r_{ij}^{\gamma_k+2}}. \tag{4}$$

As Cauchy's theorem ensures, for given initial data $(\mathbf{r}, \dot{\mathbf{r}})(t=0)$, there exists an analytic solution of (4), defined on a maximal interval $(\tilde{t}^-, \tilde{t}^+)$, namely $-\infty \le \tilde{t}^- < 0 < \tilde{t}^+ \le +\infty$, and this solution is unique. In case $\tilde{t}^\pm = \pm\infty$, the solution is regular; the situations $\tilde{t}^- > -\infty$, $\tilde{t}^+ < +\infty$ define singularities.

Remark 1. It is easy to check that equations (4) are time-reversible (see, e.g., Mioc and Barbosu 2003; Mioc and Pérez-Chavela 2008), therefore we may confine our study to the interval $[0, \tilde{t}^+)$.

Let us recall that the moment of inertia of the system is defined by

$$2\,J(\mathbf{r}) = \sum_{i=1}^{n} m_i \,|\,\mathbf{r}_i\,|^2 \;, \tag{5}$$

and let $M = \sum_{i=1}^{n} m_i$ be the total mass of the system. From now on, we denote simply $J(\mathbf{r}) = J(\mathbf{r}(t))$ by J .

PROPOSITION 1. *In the n-body problem associated to Fock's field, the following relation holds:*

$$\sum_{1 \le i < j \le n} m_i m_j r_{ij}^2 = 2MJ \;. \tag{6}$$

Proof. Starting from the expansion of the expression $\sum_{i=1}^{n} m_i\,(\mathbf{r}_i - \mathbf{r}_j)^2$, resorting to the momentum integral $\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i = \mathbf{0} = (0,0,0) \in \mathbf{R}^3$ (corresponding to origin fixed in the common mass centre of the *n*-body system), and following literally the argument exposed in Wintner's (1941) textbook, the equality (6) results easily.□

3. IMPOSSIBILITY OF SIMULTANEOUS TOTAL COLLAPSE IN INFINITE TIME

Now we are in the position to tackle the simultaneous total collapse. This means that there exists an instant $\tilde{t}^{\,+} > 0$ such that $r_{ij} \to 0$, $\forall i, j, 1 \le i < j \le n$, for $t \to \tilde{t}^{\,+}$ (observe that, according to Remark 1, we consider only the positive semiaxis of time). Some necessary results can be stated as

THEOREM 1. *The necessary and sufficient condition for the simultaneous total collapse in the n-body problem under Fock's potential is*

$$J \to 0 \quad when \quad t \to \tilde{t}^{\,+} \,. \tag{7}$$

Proof. Equations (5) and (6) lead straightforwardly to the result.□

LEMMA 1. *For simultaneous total collapse in the n-body problem under Fock's potential,* $\ddot{J} \to -\infty$ *when* $t \to \tilde{t}^{\,+}$.

Proof. We already proved (Mioc et al. 2011) that, in a much more general model, which includes Fock's potential as a particular case, the Lagrange–Jacobi relation holds. Within Fock's model this relation reads

$$\ddot{J} = \sum_{k=1}^{4}(2-\gamma_k)U_k + 2h\,. \tag{8}$$

where $h \in \mathbf{R}$ is the constant of energy. Taking into account (3) and the values of γ_k, $k = \overline{1,4}$, equation (8) becomes

$$\ddot{J} = \sum_{1\le i<j\le n}\left(\frac{A_{1,ij}}{r_{ij}} - \frac{A_{3,ij}}{r_{ij}^3} - 2\frac{A_{4,ij}}{r_{ij}^4}\right) + 2h\,, \tag{9}$$

which obviously tends to $-\infty$ when $r_{ij} \to 0$. The lemma is proved.□

Finally, we state the main result of this paper

THEOREM 2. *Every solution of equations* (4) *that leads to simultaneous total collapse reaches this limit in finite time.*

Proof. Let us suppose that there exists a solution of equations (4) that leads to the simultaneous total collision for $t \to \tilde{t}^+ = +\infty$. By virtue of Lemma 1, $\ddot{J} \to -\infty$ when $t \to \tilde{t}^+$. This means that there exist an instant $t_1 \ge 0$ and a real constant $K_1 < 0$ such that $\ddot{J} < K_1$, $\forall t > t_1$. Integrating this inequality twice between t_1 and *t*, we obtain

$$J < at^2 + bt + c\,, \tag{10}$$

where $a = K_1/2$, whereas b and c are not important for $t \to +\infty$. Since $K_1 < 0$, $J \to -\infty$ for $t \to +\infty$. But this contradicts Theorem 1. This completes the proof.□

Remark 2. By virtue of Remark 1, we proved this result only for the case $t \in [0, \tilde{t}^+)$, namely for a simultaneous total collapse in the future. Using a wholly analogous reasoning, the result remains valid for $t \in (\tilde{t}^-, 0]$, namely for a simultaneous total collapse in the past.

Remark 3. From the standpoint of astronomy (and especially cosmology), the fact that the simultaneous total collapse in the past cannot reach $\tilde{t}^- = -\infty$ is particularly important. This means that, within the framework of this more general model of Fock's field, the initial ejection (Big Bang) occurred in a finite past, like in the classical Newtonian model.

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