LAGRANGE–JACOBI AND SUNDMAN RELATIONS IN QUASIHOMOGENEOUS MODELS

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Abstract. We tackle the *n*-body problem attached to quasihomogeneous potentials. We prove that relations analogous to the Lagrange–Jacobi equality and to Sundman's inequalities exist within the framework of this much more general model, too. These results are valid for fields of different nature, not necessarily gravitational.

Key words: celestial mechanics – *n*-body problem – quasihomogeneous fields.

1. INTRODUCTION

The quasihomogeneous models in classical and celestial mechanics are more than three centuries old. Newton was the first to study such a model in his *Principia* (Book I, Article IX, Proposition XLIV, Theorem XIV, Corollary 2). He considered a gravitational force deriving from a potential of the form $A/r + B/r^2$. After Newton, such a potential was considered by Clairaut (see, e.g., Diacu et al. 1995, 2000; Delgado et al. 1996; Mioc and Stoica 1997).

Quasihomogeneous potentials regardless to the nature of the forces they generate were approached by Diacu (1996) (expansion with two terms) or Mioc and Stavinschi (2002) (expansion with N terms). In these models the powers of r are not necessarily integers.

In this paper we shall tackle a much more general model of quasihomogeneous potential. This potential covers all the above quoted models and many others.

DEFINITION 1.1. *We will call quasihomogeneous a potential having the form of a sum of homogeneous potentials:*

$$U(\mathbf{r}) = \sum_{k=1}^{N} U_k(\mathbf{r}) = \sum_{k=1}^{N} \frac{A_k}{|\,\mathbf{r}\,|^{\gamma_k}}, \tag{1.1}$$

where the parameters A_k *have different analytical expressions according to the field they characterize* (*but they depend neither on* $\mathbf{r}$, *nor explicitly on time*), γ_k *are real numbers* ($\gamma_k < \gamma_{k+1}, k = \overline{1, N-1}$), *whereas* $\mathbf{r}$ *stands for the radius vector of one particle with respect to another in the force field generated by this potential*.

Remark 1.2. As far as our knowledge goes, potential (1.1) is much more general than the above quoted ones for a twofold reason: (i) γ_k may run all along the real axis; (ii) such a model allows the study of particle dynamics under hybrid forces of totally different nature.

Remark 1.3. In many applications in astronomy, the expression (1.1) represents a truncated series. However, we also consider here the case $N = \infty$ for generality (see Condition 2.1 in Section 2, and see Section 5 for examples), even if in studies of concrete situations N is finite.

Within this very general framework, our results provide a unifying viewpoint (physical and mathematical) for a lot of problems of particle dynamics (we direct again the reader to Section 5). We shall extend some results of the classical celestial mechanics to such a general potential.

In Section 2, we establish the basic equations of the *n*-body problem under the potential (1.1). We state a condition allowing the extension of the theory to the case $N = \infty$. We point out the first integrals of motion, out of which we use only the angular momentum and energy integrals.

Section 3 proves the existence of an analog of the classical Lagrange–Jacobi relation in this much more general framework, too. This relation connects the moment of inertia of the *n*-body system to the potential and constant of energy.

Section 4 proves the existence of the two classical Sundman-type inequalities in this general model. They connect the angular momentum of the system to the moment of inertia and the potential.

In Section 5 we exemplify the applicability of our general model of potential to problems of particle dynamics in the most various fields: purely gravitational (either classical or relativistic), purely nongravitational, hybrid. All of them (single or mixed) may have different physical nature. The classical results remain valid within this much more general framework.

Section 6 emphasizes what our endeavour brings new as compared to results obtained so far concerning particle dynamics in quasihomogeneous fields.

2. BASIC EQUATIONS

Let us consider a system of n interacting particles $m_i > 0,\ i = \overline{1, n}$; let $\mathbf{r}_i = (x_i, y_i, z_i) \in \mathbf{R}^3$ be their position vectors with respect to an arbitrary origin; let $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \in \mathbf{R}^{3n}$ be the configuration of the system. Let the motion of the system be ruled by a quasihomogeneous force deriving from a potential function of the form (1.1), in which

$$U_k(\mathbf{r}) = \sum_{1 \le i < j \le n} \frac{A_{k,ij}}{r_{ij}^{\gamma_k}} . \tag{2.1}$$

Here $U_k : (\mathbf{R}^{3n} \setminus \Delta) \to \mathbf{R}$ for $\gamma_k > 0$, and $U_k : \mathbf{R}^{3n} \to \mathbf{R}$ for $\gamma_k \le 0$; $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$; $\Delta = \cup_{1 \le i < j \le n} \{\mathbf{r} \mid \mathbf{r}_i = \mathbf{r}_j\}$ is the collision set, whereas $A_{k,ij} : \mathbf{R}^2 \to \mathbf{R}$, are symmetric functions (mainly of masses, but not only, as we shall see in the last section): $A_{k,ij} = A_{k,ji}$.

The dynamics of this n-body system in such a field is described by the vectorial equation

$$m_i \ddot{\mathbf{r}}_i = \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_i} = -\sum_{1 \le i < j \le n} (\mathbf{r}_i - \mathbf{r}_j) \sum_{k=1}^{N} \gamma_k \frac{A_{k,ij}}{r_{ij}^{\gamma_k + 2}} . \tag{2.2}$$

To be able to tackle the case $N = \infty$, too (see Remark 1.3), we state

CONDITION 2.1. *The series of functions* $\sum_{k=1}^{\infty} \gamma_k A_{k,ij} / r_{ij}^{\gamma_k + 2}$ (*see* (2.2) *above*) *converges uniformly on* $\mathbf{R}^{3n} \setminus \Delta$.

Remark 2.2. Because the series $\sum_{k=1}^{\infty} A_k / |\mathbf{r}|^{\gamma_k}$ is simply convergent to $U(\mathbf{r})$ and the series of derivatives $\sum_{k=1}^{\infty} \gamma_k A_{k,ij} / r_{ij}^{\gamma_k + 2}$ is uniformly convergent, then, by the Theorem of differentiation term by term of the series of functions, the series of derivatives tends to $\partial U(\mathbf{r}) / \partial \mathbf{r}_i$ and is continuous on $\mathbf{R}^{3n} \setminus \Delta$.

Remark 2.3. It is clear that, putting $\mathbf{q}_i = \mathbf{r}_i$, $\mathbf{q} = \mathbf{r}$ (the configuration vector), $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) \in \mathbf{R}^{3n}$ (the momentum vector), and defining $H(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) - U(\mathbf{q})$ as the Hamiltonian function (where T is the kinetic energy), equations (2.2) can be transposed into a canonical form.

Standard results of the theory of differential equations ensure, for given initial conditions $(\mathbf{r},\dot{\mathbf{r}})(t=0)$, the existence and uniqueness of an analytic solution of the system (2.2), defined on an interval (t^-,t^+), $t^- < 0 < t^+$. This can be analytically extended to a maximal interval $(\tilde{t}^-,\tilde{t}^+)$, $-\infty \le \tilde{t}^- \le t^- < 0 < t^+ \le \tilde{t}^+ \le +\infty$. If $\tilde{t}^{\pm} = \pm\infty$, the solution is regular; else, it encounters a singularity.

There is no difficulty to prove that there exist ten classical first integrals for the system (2.2): the integrals of momentum $\sum_{i=1}^n m_i\dot{\mathbf{r}}_i = \boldsymbol{\alpha},\ \boldsymbol{\alpha}\in\mathbf{R}^3$; the integrals of mass centre $\sum_{i=1}^n m_i\mathbf{r}_i - (\sum_{i=1}^n m_i\dot{\mathbf{r}}_i)\,t = \boldsymbol{\beta},\ \boldsymbol{\beta}\in\mathbf{R}^3$; the integrals of angular momentum

$$\sum_{i=1}^{n}(m_i\mathbf{r}_i \times \dot{\mathbf{r}}_i) = \mathbf{C}, \quad \mathbf{C}\in\mathbf{R}^3\,; \tag{2.3}$$

and the integral of energy

$$T(\dot{\mathbf{r}}) - U(\mathbf{r}) = h, \quad h\in\mathbf{R}\,, \tag{2.4}$$

where $\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{C}$ and h are integration constants.

In the last relation, the kinetic energy of the system has the expression

$$T:\mathbf{R}^{3n}\to[0,+\infty), \quad T(\dot{\mathbf{r}}) = \frac{1}{2}\sum_{i=1}^{n} m_i\,|\dot{\mathbf{r}}_i|^2\,. \tag{2.5}$$

Remark 2.4. We may fix the origin of the coordinates in the common mass centre of the *n*-body system. Performing this transformation, equations (2.2) and the integrals (2.3) and (2.4) keep their form, but in the integrals of momentum and of mass centre we shall have, without loss of generality, $\boldsymbol{\alpha} = \mathbf{0} = \boldsymbol{\beta}$, with $\mathbf{0} = (0,0,0)\in\mathbf{R}^3$. It is easy to see that, with this particular origin, the results as regards the motion remain the same.

3. LAGRANGE–JACOBI RELATION

Consider the *n*-body system of interacting particles $m_i > 0,\ i = \overline{1,n}$, and let $\mathbf{r}_i = (x_i, y_i, z_i)\in\mathbf{R}^3$ be their position vectors with respect to the origin $\mathbf{0} = (0,0,0)\in\mathbf{R}^3$. The moment of inertia $J(\mathbf{r})$ of the system is defined by

$$J(\mathbf{r}) = \frac{1}{2}\sum_{i=1}^{n} m_i \, |\mathbf{r}_i|^2 \, . \tag{3.1}$$

Remark 3.1. It is obvious from (3.1) that the moment of inertia represents a physical measure of the distribution (scattering) of the bodies (particles) in space.

THEOREM 3.2. *In the n-body problem associated to a quasihomogeneous field, the following relation holds:*

$$\ddot{J}(\mathbf{r}) = \sum_{k=1}^{N} (2-\gamma_k) U_k(\mathbf{r}) + 2h \, , \tag{3.2}$$

where $\ddot{J}(\mathbf{r})$ *is the second derivative of* $[J(\mathbf{r})]$ *with respect to the time.*

Proof. Differentiating the expression (3.1) of the moment of inertia $J(\mathbf{r})$ with respect to the time, we get

$$\dot{J}(\mathbf{r}) = \sum_{i=1}^{n} m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i \; . \tag{3.3}$$

The time-differentiation of (3.3) leads to

$$\ddot{J}(\mathbf{r}) = \sum_{i=1}^{n} m_i (|\dot{\mathbf{r}}_i|^2 + \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i) \, . \tag{3.4}$$

By virtue of (1.1), (2.1), and (2.2), the following relation results easily:

$$\sum_{i=1}^{n} \mathbf{r}_i \cdot \frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_i} = -\sum_{k=1}^{N} \gamma_k U_k(\mathbf{r}) \, , \tag{3.5}$$

or, taking into account (2.2) once again,

$$\sum_{i=1}^{n} m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = -\sum_{k=1}^{N} \gamma_k U_k(\mathbf{r}) \, . \tag{3.6}$$

On the other hand, formulae (1.1), (2.1), (2.4), and (2.5) lead immediately to

$$\sum_{i=1}^{n} m_i |\dot{\mathbf{r}}_i|^2 = 2\sum_{k=1}^{N} U_k(\mathbf{r}) + 2h \, . \tag{3.7}$$

Adding (3.6) and (3.7) together, and plugging the resulting expression in (3.4), the equality (3.2) follows straightforwardly. This completes the proof. □

Remark 3.3. The equality (3.2) is nothing else but the Lagrange–Jacobi relation transposed for the much more general quasihomogeneous model.

Remark 3.4. The proof of Theorem 3.2 follows that given by Wintner (1941) for the Newtonian model. The respective arguments have also been used by Mioc and Stoica (1996) to prove an analogous theorem for a sum of two homogeneous potentials.

4. SUNDMAN-TYPE INEQUALITIES

Within the Newtonian model, the inequalities of Sundman connect the moment of inertia and the angular momentum (of course, under the respective potential). We shall prove that inequalities of this type hold within the quasihomogeneous models, too. To begin, we state

THEOREM 4.1. *In a quasihomogeneous field, the following inequality holds:*

$$|\mathbf{C}|^2 \le 2J(\mathbf{r})[\ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k U_k(\mathbf{r})]. \tag{4.1}$$

Proof. Recall the well-known inequality $|\sum_{i=1}^{n} \mathbf{x}_i| \le \sum_{i=1}^{n} |\mathbf{x}_i|$. Using this inequality in the case of formula (2.3), we get

$$|\mathbf{C}| \le \sum_{i=1}^{n} m_i |\mathbf{r}_i \times \dot{\mathbf{r}}_i|. \tag{4.2}$$

Squaring (4.2), and recalling that $|\mathbf{x} \times \mathbf{y}| \le |\mathbf{x}| \times |\mathbf{y}|$, we obtain

$$|\mathbf{C}|^2 \le [\sum_{i=1}^{n} (\sqrt{m_i}\, |\mathbf{r}_i|)(\sqrt{m_i}\, |\dot{\mathbf{r}}_i|)]^2. \tag{4.3}$$

Here we have to resort to the classical inequality

$$(\sum_{i=1}^{n} \mathbf{x}_i \cdot \mathbf{y}_i)^2 \le (\sum_{i=1}^{n} |\mathbf{x}_i|^2)(\sum_{i=1}^{n} |\mathbf{y}_i|^2). \tag{4.4}$$

Applying (4.4) to (4.3), it results

$$|\mathbf{C}|^2 \le (\sum_{i=1}^{n} m_i\, |\mathbf{r}_i\,|^2\,)(\sum_{i=1}^{n} m_i\, |\dot{\mathbf{r}}_i\,|^2\,)\,. \tag{4.5}$$

Resorting to (2.5) and (3.1), it follows immediately

$$|\mathbf{C}|^2 \le 4\,J(\mathbf{r})\,T(\dot{\mathbf{r}})\,. \tag{4.6}$$

By the energy integral (2.4) and the relation (3.2) from Theorem 3.2, one easily obtains

$$2T(\dot{\mathbf{r}}) = \ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k\, U_k(\mathbf{r})\,, \tag{4.7}$$

which, replaced in (4.6), leads to the inequality (4.1). This completes the proof. □

This result can be refined in the form of

THEOREM 4.2. *In a quasihomogeneous field, an inequality stronger than* (4.1) *holds:*

$$|\mathbf{C}|^2 \le 2\,J(\mathbf{r})[\ddot{J}(\mathbf{r}) + \sum_{k=1}^{N} \gamma_k\, U_k(\mathbf{r})] - [\dot{J}(\mathbf{r})]^2\,. \tag{4.8}$$

Proof. Resorting again to $|\sum_{i=1}^{n}\mathbf{x}_i\,| \le \sum_{i=1}^{n}|\mathbf{x}_i\,|$, and applying this to (3.3), we obtain

$$|\dot{J}(\mathbf{r})| \le \sum_{i=1}^{n} m_i\, |\mathbf{r}_i\,||\dot{\mathbf{r}}_i\,|\,. \tag{4.9}$$

Squaring (4.9), and using (4.4), we get

$$[\dot{J}(\mathbf{r})]^2 \le (\sum_{i=1}^{n} m_i\, |\mathbf{r}_i\,|^2\,)(\sum_{i=1}^{n} m_i\, |\dot{\mathbf{r}}_i\,|^2\,)\,, \tag{4.10}$$

relation that, taking into account the definition (3.1) of the moment of inertia, may be written in the form

$$[\dot{J}(\mathbf{r})]^2 \le 2\,J(\mathbf{r})\sum_{i=1}^{n}[m_i(\mathbf{r}_i \cdot \dot{\mathbf{r}}_i\,)^2 / |\mathbf{r}_i\,|^2\,]\,. \tag{4.11}$$

Now, rewriting (4.2) as

$$|\mathbf{C}| \leq \sum_{i=1}^{n} (\sqrt{m_i}\ |\mathbf{r}_i|)(\sqrt{m_i}\ |\mathbf{r}_i \times \dot{\mathbf{r}}_i| / |\mathbf{r}_i|), \tag{4.12}$$

squaring, and using (4.4) again, then considering (3.1), we obtain

$$|\mathbf{C}|^2 \leq 2\,J(\mathbf{r}) \sum_{i=1}^{n} (m_i\ |\mathbf{r}_i \times \dot{\mathbf{r}}_i|^2 / |\mathbf{r}_i|^2). \tag{4.13}$$

Adding (4.11) and (4.13) together, then using the well-known relation $|\mathbf{x} \times \mathbf{y}|^2 + |\mathbf{x} \cdot \mathbf{y}|^2 = |\mathbf{x}|^2 |\mathbf{y}|^2$, and taking into account the definition (2.5), we easily get the inequality

$$|\mathbf{C}|^2 + [\dot{J}(\mathbf{r})]^2 \leq 4\,J(\mathbf{r})\,T(\dot{\mathbf{r}}). \tag{4.14}$$

Finally, substituting the expression (4.7) of $T(\dot{\mathbf{r}})$ in (4.14), the inequality (4.8) is obtained. This completes the proof. □

Remark 4.3. The inequalities (4.1) and (4.8) are analogous to those established by Sundman (see, e.g., Wintner 1941) for the classical Newtonian potential.

5. APPLICABILITY

In this section we shall point out some concrete fields covered by the quasihomogeneous model featured by Definition 1.1. Our previous results are obviously valid within all these fields. Of course, the parameters A_k differ from model to model; they will not be specified here.

5.1. Classical gravitational models:

– Manev's model, with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$ (e.g., Maneff 1924, 1925, 1930a,b; Diacu 1993; Mioc and Stoica 1995a,b; etc.).

– The J_2 problem (main problem of satellite dynamics), with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 3$ (the bibliography is huge and easily available).

– The zonal satellite problem, with $N \to \infty$, $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_k = k$ $(k \geq 3)$ (e.g., Mioc and Stavinschi 1998a,b; etc.).

5.2. Relativistic gravitational models:

– Schwarzschild model, with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 3$ (the bibliography is huge and easily available).

– Fock's model, with $N = 4$, $\gamma_k = k, k = \overline{1,4}$ (e.g., Fock 1959; Mioc and Pérez-Chavela 2008; etc.).

– Schwarzschild – de Sitter model, with $N = 3$, $\gamma_1 = -2$, $\gamma_2 = 1$, $\gamma_3 = 3$ (e.g., Blaga and Mioc 1992; Mioc and Pérez-Chavela 2010; etc.).

5.3. Classical nongravitational models:

– The diffuse re-emitted radiation pressure, with $N \to \infty$, γ_k integers (e.g., Mioc and Radu 1982; etc.), or the infrared re-emitted radiation pressure.

– The Lennard-Jones model with $N = 2$, $\gamma_1 = 6$, $\gamma_2 = 12$ (Lennard-Jones 1931; Mioc et al. 2008a,b; etc.).

– A planetary magnetic field, with $N \to \infty$, γ_k integers (the bibliography is huge and easily available).

5.4. Mixed classical models:

– The photogravitational model (gravitation plus radiation) with a non-Newtonian gravitational force.

– The gravito-elastic model, with $N = 2$, $\gamma_1 = -2$, $\gamma_2 = 1$ (e.g., Mioc and Stavinschi 1999).

– Models from atomic physics: the potential energy of an outward electron in the field of the nucleus, with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$ (e.g., Sommerfeld 1951; Belenkii 1981).

5.5. Relativistic mixed models:

– The Reissner–Nordström model, with $N = 2$, $\gamma_1 = 1$, $\gamma_2 = 2$.

Of course, the models covered by our quasihomogeneous potential are much more numerous than the ones presented above.

6. SUMMARY AND NEW RESULTS

In this last section we shall summarize what we have done in this paper and what we brought new as compared to previous results in this domain.

We defined a very general model, in which the interaction between particles is mainly gravitational, but a lot of other forces are allowed.

We proved the existence of an analog of the classical Lagrange–Jacobi relation in this much more general framework.

We proved the existence of the two classical Sundman-type inequalities in this general model.

We exemplified the applicability of our general model of potential to problems of particle dynamics in the most various fields: purely gravitational (either classical or relativistic), purely nongravitational, hybrid. All of them (single or mixed) may have different physical nature. The classical results remain valid within this framework.

As to new results, we mention that we generalized the previous results to the case $N = \infty$, not tackled in this kind of analyses yet.

We also generalized the previous results by taking into account negative values of γ_k (so far only positive γ_k were considered). In this way, models as the Schwarzschild – de Sitter one or the gravito-elastic one can join these general results, too.

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