APPROXIMATE MODELS OF STELLAR ORBITS

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Abstract. On galactic scales, stars can be considered as point sources that move virtually without collisions on orbits determined by their common gravitational potential. Most of published studies start with classical epicycle theory and try to correct it. Unfortunately, the most accurate orbit approximation leads to non-analytical $R(t)$. From this approximation, accurate estimates can be obtained for action integrals of galactic orbits. For this work, I consider the action integral and manipulate it to obtain the approximate solutions of stellar orbits.

Key words: stellar dynamics – celestial mechanics – galaxies.

1. INTRODUCTION

Stars travel within and around the Galaxy, and galaxies orbit within their groups and clusters, under the force of gravity. Stars are so much denser than the interstellar gas through which they move that neither gas pressure nor the forces from embedded magnetic fields can deflect them from their trajectories. If we know how mass is distributed, we can find the resulting gravitational force, and from this we can calculate how the positions and velocities of stars and galaxies will change over time.

We can also use the stellar motions to tell us where the mass is. As we know, much of the matter in the Milky Way cannot be seen directly. Its radiation may be absorbed, as happens for the visible light of stars in the dusty disk. Some material simply emits too weakly: dense clouds of cold gas do not show up easily in radio-telescope maps. The "infamous" dark matter still remains invisibly mysterious. But, since the orbits of stars take them through different regions of the galaxies they inhabit, their motions at the time we observe them have been affected by the gravitational fields through which they have travelled earlier. So we can use the equations for motion under gravity to infer from observed motions how mass is distributed in those parts of galaxies that we cannot see directly.

Usually we can consider the stars as point masses, because their sizes are small compared with the distances between them. Since galaxies contain anywhere between a million stars and 10^{12} of them, we usually want to look at the average motion of many stars, rather than following the individual orbit of each one. We prove the *virial theorem*, relating average stellar speeds to the depth of the gravitational potential well in which they move. Orbital dynamics and the virial theorem are our tools to find masses of star clusters and galaxies.

The gravitational potential of a galaxy or star cluster can be considered as the sum of a smooth component, the average over a region containing many stars, and the very deep potential well around each individual star. We will see that the motion of stars within a galaxy is determined almost entirely by the smooth part of the force. Two-body encounters, transferring energy between individual stars, can be important within dense star clusters. We discuss how these encounters change the cluster's structure, eventually causing its dispersion or "evaporation". The epicycle theory is a way to simplify the calculation of motions of stars that follow very nearly circular orbits within a galaxy's disk.

2. MOTION UNDER GRAVITY

Newton's law of gravity tells us that a point mass M attracts a second mass m separated from it by distance $\mathbf{r}$, causing the velocity $\mathbf{v}$ of m to change according to

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r}\,, \tag{1}$$

where G is the Newtonian gravitational constant. In a cluster of N stars of masses m_α ($\alpha = \overline{1, N}$), at positions $\mathbf{x}_\alpha$, we can add the forces on star α from all the other stars:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_\alpha \mathbf{v}_\alpha) = -\sum_{\substack{\beta \\ \alpha\neq\beta}} \frac{Gm_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta)\,. \tag{2}$$

The mass m_α cancels out of this equation, so the acceleration $\mathrm{d}\mathbf{v}_\alpha/\mathrm{d}t$ is independent of the star's mass: light and heavy objects fall equally fast. This is the *principle of equivalence* between gravitational and inertial mass, which is the basis for the general relativity theory.We can write the force from the cluster on a star of mass m

at position $\mathbf{x}$ as the gradient of the *gravitational potential* $\Phi(\mathbf{x})$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = -m\nabla\Phi(\mathbf{x}), \qquad \Phi(\mathbf{x}) - \sum_{\alpha} \frac{Gm_{\alpha}}{|\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}|}(\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}) \quad \text{for} \quad \mathbf{x} \neq \mathbf{x}_{\alpha}\,, \tag{3}$$

where we have chosen an arbitrary integration constant so that $\Phi(\mathbf{x}) \to 0$ at large distances. If we think of a continuous distribution of matter in a galaxy or star cluster, the potential at point $\mathbf{x}$ is given by an integral over the density $\rho(\mathbf{x}')$ at all other points:

$$\Phi(x) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'\,, \tag{4}$$

and the force per unit mass is

$$\mathbf{F(x)} = -\nabla\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}\,\mathrm{d}^3\mathbf{x}'. \tag{5}$$

The integral relation of equation (4) can be turned into a differential equation. Applying ∇^2 to both sides, we have

$$\nabla^2\Phi(\mathbf{x}) = -\int G\rho(\mathbf{x}')\nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}'. \tag{6}$$

In three dimensions, differentiating with respect to the variable $\mathbf{x}$ gives, for $\mathbf{x} \neq \mathbf{x}'$:

$$\nabla\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}\,, \qquad \nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right) = 0\,. \tag{7}$$

So the integrand on the right-hand side of equation (6) is zero outside a small sphere $\mathrm{S} \in (\mathbf{x})$ of radius centered on $\mathbf{x}$. If we consider that the density ρ is almost constant inside the sphere, we have

$$\nabla^2\Phi(\mathbf{x}) \approx -G\rho(\mathbf{x}) \int_{\mathrm{S}\in(\mathbf{x})} \nabla^2\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)\mathrm{d}^3\mathbf{x}' = -G\rho(\mathbf{x}) \int_{\mathrm{S}\in(\mathbf{x})} \nabla^2_{\mathbf{x}'}\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|}\right)\mathrm{d}V'\,. \tag{8}$$

In the last step, $\nabla^2_{\mathbf{x}'}$ means that the derivative is taken with respect to the variable $\mathbf{x}'$, instead of $\mathbf{x}$.

Now we can use the *divergence theorem*: for any smooth-enough function f , the volume integral of $\nabla^2_{\mathbf{x}'} f$ over the interior of any volume is equal to the integral of $\nabla^2_{\mathbf{x}'} f \cdot \mathrm{d}S'$ over the surface. We also have $\nabla^2_{\mathbf{x}'} f = -\nabla f$ for any function $f\,(\mathbf{x}-\mathbf{x}')$. Setting $f = 1/|\,\mathbf{x}-\mathbf{x}'\,|$, equation (7) tells us that, on the surface of the sphere $S \in (\mathbf{x})$, the gradient $\nabla_{\mathbf{x}'} f$ is a vector of length ε^{-2} pointing in toward the point $\mathbf{x}$. The surface area is $4\pi\varepsilon^2$, so the integral of $\nabla^2_{\mathbf{x}'} f \cdot \mathrm{d}S'$ in equation (8) is -4π. We have Poisson's equation:

$$\nabla_{\mathbf{x}'}\Phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x})\,. \tag{9}$$

This can be a more convenient relationship between the potential $\Phi(\mathbf{x})$ and the corresponding density than the integral in equation (4). To choose an approximation for the density $\rho(\mathbf{x})$ of a star cluster or galaxy, we can select a mathematically convenient form for the potential $\Phi(\mathbf{x})$, and then calculate the corresponding density.

We must take care that $\rho(\mathbf{x}) \geq 0$ everywhere for our chosen potential; various apparently friendly potentials turn out to imply $\rho(\mathbf{x}) < 0$ in some places. The problems below deal with some commonly used potentials.

For a spherical galaxy or star cluster, Isaac Newton proved two useful theorems about the gravitational field. The first states that the gravitational force inside a spherical shell of uniform density is zero. In Fig. 1, the star at S experiences a gravitational pull from the material at A within a narrow cone of solid angle $\Delta\Omega$, and a force in the opposite direction from mass within the same cone at B. By symmetry, the line AB makes the same angle with the normal OA to the surface at A as it does with OB at B. Thus the ratio of the mass enclosed is just $(\mathrm{SA}/\mathrm{SB})^2$; by the inverse-square law, the forces are exactly equal, and cancel each other out. Thus there is no force on the star, and the potential $\Phi(\mathbf{x})$ must be constant within the shell.

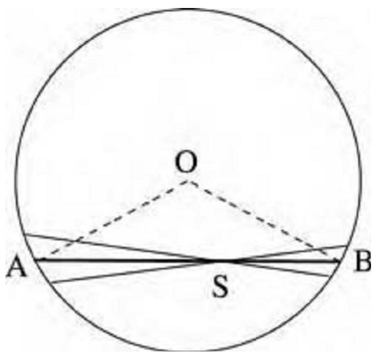

Fig. 1 – The gravitational force inside a uniform hollow sphere with its center at O.

The second theorem says that, outside any spherically symmetric object, the gravitational force is the same as if all its mass had been concentrated at the center. If we can show that for a uniform spherical shell, it must be true for any spherically symmetric object built from those shells. To find the potential $\Phi(\mathbf{x})$ at a point P lying outside a uniform spherical shell of mass M and radius a, at distance r from the center, we can add the contributions $\Delta\Phi$ from small patches of the shell. On the left of Fig. 2, the mass in a narrow cone of opening solid angle $\Delta\Omega$ around Q' contributes

$$\Delta[x(\mathrm{P})] = -\frac{GM}{|x(\mathrm{P}) - x(\mathrm{Q}')|}\frac{\Delta\Omega}{4\pi}. \tag{10}$$

Now think of the potential Φ' at point P', lying at distance a from the center inside a sphere of the same mass M, but now with radius r. On the right of Fig. 2, we see that the contribution $\Delta\Phi'$ from material in the same cone, which cuts the larger sphere at Q, is

$$\Delta\Phi'[\mathbf{x}(\mathrm{P}')] = -\frac{GM}{|\mathbf{x}(\mathrm{P}') - \mathbf{x}(\mathrm{Q}')|}\frac{\Delta\Omega}{4\pi}. \tag{11}$$

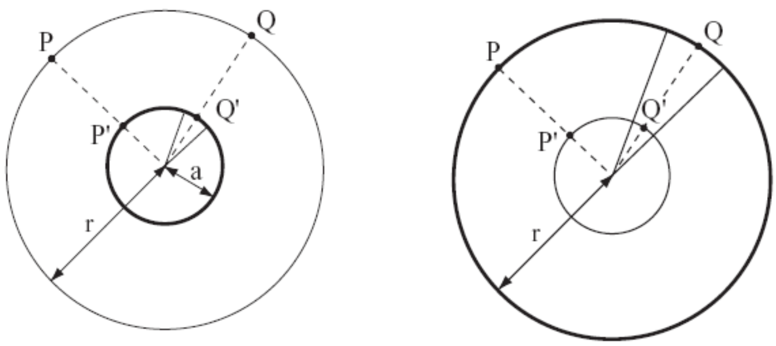

Fig. 2 – The gravitational potential outside a uniform spherical shell.

But, because $\mathrm{PQ}' = \mathrm{P}'\mathrm{Q}$, this is equal to $\Delta\Phi[\mathbf{x}(\mathrm{P})]$. So, when we integrate over the whole sphere,

$$\Phi[\mathbf{x}(\mathrm{P})] = \Phi'[\mathbf{x}(\mathrm{P}')] = \Phi'[x(\mathrm{P}')] = \Phi'[x = 0] = -GM/|r|, \tag{12}$$

the potential and force at P are exactly the same as if all the mass of the sphere with radius a had been concentrated at its center.

These two theorems tell us that, within any spherical object with density $\rho(r)$, the gravitational force toward the center is just the sum of the inward forces from all the matter inside that radius. The acceleration V^2/r of a star that moves with speed $V(r)$ in an orbit of radius r about the center must be provided by the inward gravitational force $-F_\mathrm{r}(r)$. So, if $M(<r)$ is the mass within radius r, we have

$$\frac{V^2(r)}{r} = -F_\mathrm{r}(r) = -\frac{GM(<r)}{r^2} \tag{13}$$

whenever we can find gas or stars in nearly-circular orbit within a galaxy, and this is the simplest and most reliable way to estimate the mass.

For a point mass, we have $V(r) \propto r^{-1/2}$; in a spherical galaxy the rotation speed can never fall more rapidly than this. The potential $\Phi(r)$ is

$$\Phi(r) = \left[\frac{GM(<r)}{r} + 4\pi G\int_r^\infty \rho(r')r'\,\mathrm{d}r'\right]. \tag{14}$$

We see that $\Phi(r)$ is *not* equal to $-GM(<r)/r$, unless the whole mass lies within the radius r. But equation (4) implies that, for a great distance from any system with finite mass M_tot,

$$\Phi(\mathbf{x}) \to -\frac{GM_\mathrm{tot}}{|\mathbf{x}|}\,. \tag{15}$$

When finding the orbit of a single star moving through a galaxy, we will see in Section 3 that we can usually ignore the effect which that star has in attracting all the other stars, and thus changing the gravitational potential. If we have a static mass distribution, the potential at position $\mathbf{x}$ does not depend on time. Then as the star moves with velocity $\mathbf{v}$, the potential $\Phi(\mathbf{x})$ at its location changes according to $\mathrm{d}\Phi/\mathrm{d}t = \mathbf{v}\cdot\nabla\Phi(\mathbf{x})$. Taking the scalar product of equation (3) with $\mathbf{v}$, we have

$$\mathbf{v}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) + m\mathbf{v}\cdot\Phi(\mathbf{x}) = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x})\right]. \tag{16}$$

Thus

$$\zeta \equiv m\mathbf{v}^2/2 + m\Phi(\mathbf{x}) = \text{const. (along the orbit)}\,. \tag{17}$$

The *star's energy* ζ is the sum of its *kinetic energy* $K\zeta = m\mathbf{v}^2/2$ and the *potential energy* $P\zeta = m\Phi(\mathbf{x})$. The kinetic energy cannot be negative, and equation (15) tells us, for an isolated galaxy or star cluster, that $\Phi(\mathbf{x}) \to 0$. So a star that lies at position $\mathbf{x}$ can escape only if it has $\zeta > 0$; it must move faster than the local *escape speed* v_{e}, given by

$$v_{\mathrm{e}}^2 = -2\Phi(\mathbf{x}). \tag{18}$$

In a cluster of stars, the gravitational potential will change as the stars move: $\Phi = \Phi(\mathbf{x}, t)$. The energy of each star is no longer conserved, only the total in case of the cluster as a whole. To show this, we take the scalar product of equation (2) with $\mathbf{v}_\alpha$, and sum over all the component stars. The left-hand side gives the derivative of the total kinetic energy $K\zeta$:

$$\sum_\alpha \mathbf{v}_\alpha \cdot \frac{\mathrm{d}}{\mathrm{d}t}(m_\alpha \mathbf{v}_\alpha) = \frac{\mathrm{d}}{\mathrm{d}t} K\zeta = -\sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{G m_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta)\cdot \mathbf{v}_\alpha. \tag{19}$$

But we could have started with the equation for the force on star β, and taken the scalar product with $\mathbf{v}_\beta$ to find

$$\frac{1}{2}\sum_\beta \frac{\mathrm{d}}{\mathrm{d}t}(m_\beta \mathbf{v}_\beta \cdot \mathbf{v}_\beta) = -\sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{G m_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|^3}(\mathbf{x}_\beta - \mathbf{x}_\alpha)\cdot \mathbf{v}_\beta. \tag{20}$$

Adding the right-hand sides of the last two equations, we get

$$-\sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{G m_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta)\cdot (\mathbf{v}_\alpha - \mathbf{v}_\beta) = \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{G m_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|}\right). \tag{21}$$

The cluster's potential energy $P\zeta$ is the sum of contributions from pairs of stars:

$$P\zeta = -\frac{1}{2}\sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \frac{G m_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|^3} = \frac{1}{2}\sum_\alpha m_\alpha \Phi(\mathbf{x}_\alpha) \quad \text{or} \quad \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\,\mathrm{d}V, \tag{22}$$

dividing by two (each pair contributes only with one term to the sum). On adding equations (19) and (20), we see that

$$2\frac{\mathrm{d}}{\mathrm{d}t}\left[K\zeta - \frac{1}{2}\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}\frac{Gm_\alpha m_\beta}{\left|\mathbf{x}_\alpha - \mathbf{x}_\beta\right|}\right] = 0. \tag{23}$$

Thus the total energy $\zeta = K\zeta + P\zeta$ of the cluster is constant.

3. TWO-BODY RELAXATION

While a star moves in the smoothed potential of a star cluster, equation (17) shows us that its orbit does not depend on it, or it is heavy or light (only on its position or velocity). If the smoothed potential $\Phi(\mathbf{x})$ does not change with time, the energy of the star remains constant. By contrast, two-body 'collisions' allow two stars to exchange energy and momentum in a way that depends on both their masses; this is known as the *two-body relaxation*.

Just as for the air molecules in a room, the exchanges on average will shift the velocities of the stars toward the most probable way of sharing the available energy: this is a Maxwellian distribution. The fraction f of stars with velocities v, between v and $v + \Delta v$ is given by $f_\mathrm{M}(\zeta)\cdot 4\pi v^2 \cdot \Delta v$, where

$$f_\mathrm{M}(\zeta) \propto \exp\left[\frac{-\zeta}{k_\mathrm{B}T}\right] = \exp\left\{-\left[m\Phi(\mathbf{x}) + \frac{mv^2}{2}\right]/(k_\mathrm{B}T)\right\}, \tag{24}$$

and k_B is Boltzmann's constant. The 'temperature' T depends on the energy of the system: it is higher when the stars are moving faster. The problem below shows that, for stars of mass m, T is related to the average of the squared velocities by

$$m\left\langle \mathbf{v}^2(\mathbf{x})\right\rangle/2 = 3k_\mathrm{B}T/2\,. \tag{25}$$

Two-body relaxation, in Maxwellian form, causes stars to *evaporate* from the cluster. The distribution $f_\mathrm{M}(\zeta)$ includes a small number of stars with arbitrarily high energy; but any stars moving faster than the escape speed v_e given by equation (18) are not bound to the cluster and will escape. In a cluster of N stars with masses m_α at positions $\mathbf{x}_\alpha$, equation (22) tells that the average kinetic energy needed for escape is

$$\left\langle\frac{1}{2}mv_\mathrm{e}^2(\mathbf{x})\right\rangle = -\frac{1}{N}\sum_\alpha m_\alpha\Phi(\mathbf{x}_\alpha) = -\frac{2}{N}P\zeta = \frac{4}{N}K\zeta\,, \tag{26}$$

where $P\zeta$ and $K\zeta$ are the potential and kinetic energy of the cluster as a whole. The average kinetic energy needed for escape is just four times the average for each star, or $6k_\mathrm{B}T$, so the fraction of escaping stars in the Maxwellian distribution f_M is

$$\int_{\sqrt{12k_BT/m}}^{\infty} f_M(\zeta)v^2\,\mathrm{d}v \Big/ \int_0^{\infty} f_M(\zeta)v^2\,\mathrm{d}v = 0.007353 \approx \frac{1}{136}\,. \tag{27}$$

These stars leave the cluster; after a further time t_relax, and new stars are promoted above by the escape energy. The cluster loses a substantial fraction of its stars over an *evaporation time*

$$t_\mathrm{evap} \approx 136 t_\mathrm{relax}\,. \tag{28}$$

In the observed globular clusters, t_evap is longer than the age of the Universe; any clusters with very short evaporation times presumably dissolved before we could observe them. For open clusters t_evap is only a few Gyr. In practice, these clusters fall apart even more rapidly, since evaporation is helped along by the repeated gravitational tugs from the spiral arms and from giant clouds of molecular gas in the disk.

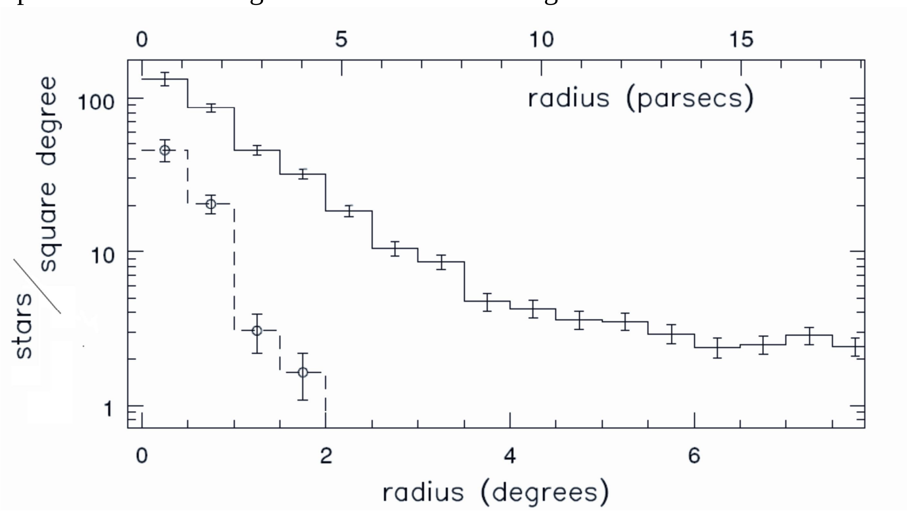

Fig. 3 – In the Pleiades open cluster, stars with masses above M_Sun (dashed histogram) are more concentrated toward the center than stars with $M < M_\mathrm{Sun}$ (solid histogram).

Two-body relaxation also causes mass segregation. Heavier stars congregate at the cluster center, while lighter stars are expelled toward the periphery; we see the result in Fig. 3. If initially the cluster stars are thoroughly mixed, with similar orbital speeds, the more massive stars will have larger kinetic energy. But, in a Maxwellian distribution, their kinetic energies must be equal. Thus, on average, a massive star will be moving slower after an encounter than it did before. It then sinks to an orbit of lower energy; so the cluster center fills up with stars that have too little energy to go anywhere else. But, as the cluster becomes centrally concentrated, these tightly bounded stars must move faster than those further out, increasing their tendency to give up energy.

Meanwhile, the upwardly mobile lighter stars have gained energy from their encounters, but spend it in moving out to the suburbs. Their new orbits require slower motion than before, so they have become even poorer in kinetic energy.

Mass segregation is a runaway process: the lightest stars are pushed outward into an ever-expanding diffuse outer halo, while the heavier stars form an increasingly dense core at the center. Almost all star clusters have been affected by mass segregation. The smallest and luminous stars, that carry most of the cluster's mass, are dispersed far from the center. So, we have to be careful to trace them when estimating the cluster's mass or the stellar mass function. Pairs of stars, bound in a tight binary, will effectively behave like a single more-massive star, sinking to the core. The X-ray sources in globular clusters are binaries in which a main-sequence star orbits a white dwarf or neutron star; they are all found near the cluster center.

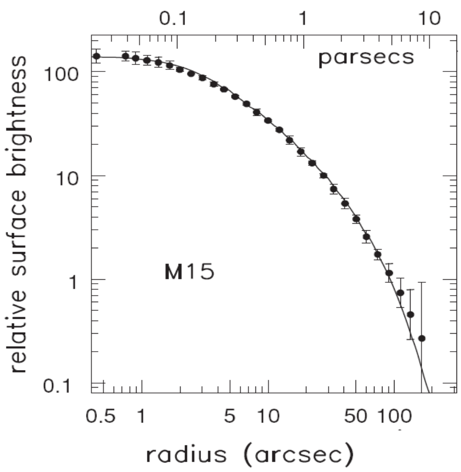

Fig. 4 – Surface brightnesses of M15 globular cluster. The constant-density core is absent, or too small to measure. The solid lines show it is in case of King model (see Binney and Tremaine 1987).

Even if all the stars in a cluster have exactly the same mass, stars on low energy orbits close to the center have higher orbital speeds than do those further out. So, the inner stars tend to lose energy, while the outer stars gain it. Over time, some stars are expelled from the cluster core into the expanding halo, whereas the remaining core contracts. The core becomes denser, while the outer parts puff up and become more diffuse. Calculations for clusters of equal-mass stars predict that, after $(12-20)t_{\text{relax}}$, the core radius shrinks to zero, as the central density increases without limit and this is core collapse. A cluster that is near this state should have a small dense core and a diffuse halo, as we see for M15 in Fig. 4. For a comparison, see also M4 in Fig. 5.

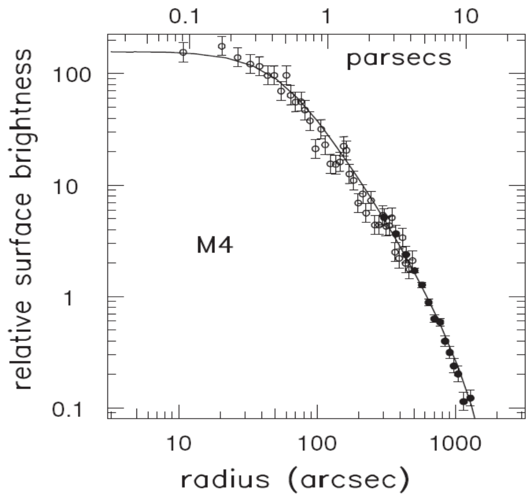

Fig. 5 – Surface brightnesses of M4 globular cluster. The surface brightness is nearly constant at small radii, dropping almost to zero at the truncation radius $r_t \approx 3000''$.

What happens to a cluster after core collapse? In the dense core, binary stars become important sources of energy. Just as two-body collisions tend to remove energy from fast-moving stars, so encounters between single stars and a tight binary pair will take energy from the binary. The energy is transferred to the single star, while the binary is forced closer. Depending on how many are present, binaries may supply so much energy to the stars around them that the core of the cluster starts to re-expand.

4. STELLAR ORBITS APPROXIMATION

In this section, a general recipe is presented for approximately solving equation of motion. I also derive a way for obtaining an approximate map between ordinary phase-space coordinates and action-angle variables. Worked examples, i.e., concrete orbit approximations, are presented in this section.

The key idea in obtaining an approximation superior to classical epicycle theory is to transform the integrals into a form which is better adapted for replacing the argument of the square root by a quadratic. Consider writing equation as follows:

$$t = \int \frac{\mathrm{d}R}{p_R} = \int \frac{\xi(R)\,\mathrm{d}R}{\xi(R)\,p_R} = \int \frac{x^n\,\mathrm{d}x}{\xi(R)\,p_R}\,, \tag{29}$$

where $\xi(R)$ is a positive definite function of R. The relation for $x(R)$ follows upon integrating $\xi(R)\,\mathrm{d}R = x^n\,\mathrm{d}x$ with integer n. We introduce the auxiliary variable η by

$$\mathrm{d}t/\mathrm{d}\eta = x^n\,, \tag{30}$$

which essentially amounts to a new time variable. For $n > 0$, equation (30) means that, for fixed $\mathrm{d}\eta$, the timestep $\mathrm{d}t$ is smaller near pericenter than near apocenter.

With (30), equation (29) becomes

$$\eta = \int Y^{-1/2}(x)\,\mathrm{d}x\,, \tag{31}$$

with

$$Y(x) \equiv 2\xi(R)[E - \Phi(R) - L^2/(2R^2)]\,. \tag{32}$$

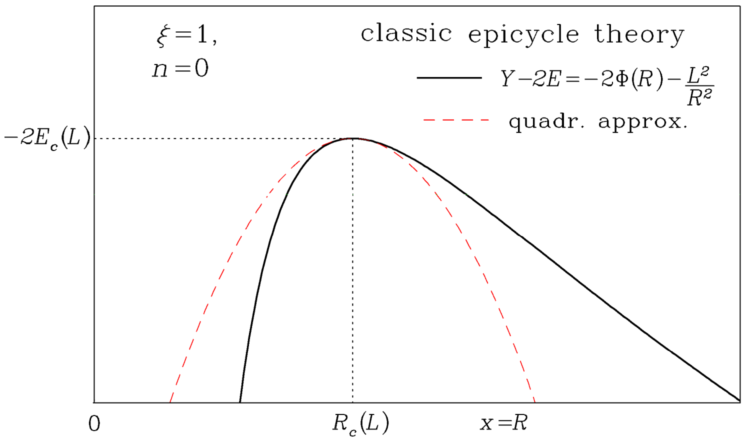

Fig. 6 – Lindblad's classical epicycle theory for the logarithmic potential. The solid line gives $Y(R)$, while the quadratic approximation is shown as broken line.

The new integral is formally identical to the original, and we might derive an approximate solution for $x(\eta)$ in the same way as Lindblad's epicycle theory is derived. Approximating

$$Y \approx X^2 - a^2(x - x_0)^2\,, \tag{33}$$

obtained, for example, via Taylor expansion, we find

$$x \approx x_0(1 - e\cos a\eta), \quad e = X/(ax_0)\,. \tag{34}$$

With this form of $x(\eta)$, equation (30) is easily solved for $t(\eta)$. However, $\eta(t)$ cannot be obtained analytically unless $n = 0$ (Kalnajs 1979; Dehnen 1999).

According to Binney and Tremaine (1987), the most useful function $\xi(R)$ in the cases of $\xi = 1$ or $\xi = R$ is:

$$Y = 2E - [2\Phi(R) + L^2/R^2]\,, \tag{35}$$

and the results, in the case of logarithmic potential, are given in Fig. 6.

5. THE AZIMUTHAL MOTION

After obtaining an expression for $R(t)$, the radial angle θ_R is easily identified as the term, linear in time, of which $R(t)$ is a 2π-periodic function. With the above formalism, the radial action is

$$J_R \equiv \frac{1}{2\pi}\oint p_R\,\mathrm{d}R = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\mathrm{d}R}{\mathrm{d}x}\right)^2\left(\frac{\mathrm{d}x}{\mathrm{d}\eta}\right)^2\frac{\mathrm{d}\eta}{\mathrm{d}t}\mathrm{d}\eta = \frac{X^2}{2\pi}\int_0^{2\pi}\frac{\sin^2 a\eta x^n\,\mathrm{d}\eta}{\xi^2(R(x))}\,, \tag{36}$$

where $x = x(\eta)$ is given in equation (31). For this integral to be soluble, $\xi(R)$ must not be too complicated when expressed as function of x.

The azimuthal motion may be obtained by inserting the approximation for $R(t)$ into equation and integrate after further approximating the integrand. However, the approximate orbits generated in this way do not necessarily produce an incompressible flow in phase-space, as Liouville's theorem demands. In other words, the part of $\varphi(t)$ which is linear in time is not, in general, the angle θ_R canonical conjugate to the angular momentum L at fixed (J_R, θ_R). When using such an approximation, spurious effects may result from this lack. Note that in particular the textbook variant of Lindblad's epicycle theory (Binney and Tremaine 1987, p. 124) suffers from this problem.

Some known approximations are given below.

5.1. CLASSICAL EPICYCLE THEORY

Lindblad's classical epicycle theory is recovered from the above recipe for $\xi = 1$ and $n = 0$ (see Shu 1969). The Taylor expansion of Y around R_L, with $\delta_R = R - R_L$, reads

$$Y = 2\Delta E - \kappa \cdot 2\delta_R^2 - \left[\frac{\mathrm{d}\kappa^2}{\mathrm{d}R} - \frac{3\kappa^2}{R}\right]_{R_L} \frac{\delta_R^3}{3} + O(\delta_R^4) \tag{37}$$

where the epicycle frequency κ is evaluated at R_L, and hence is a function of angular momentum. The large error of this approximation is related to the neglected third-order term in (37), which for stellar systems is never small, since always $\mathrm{d}\kappa^2/\mathrm{d}R < 0$ (see Dehnen 1998). The radial motion is approximated by

$$R(t) = R_L(1 - e\cos\theta_R)\,, \tag{38}$$

$$e = \sqrt{2\Delta E}\,/(R_L\kappa)\,, \tag{39}$$

$$J_R = \kappa R_L^2 e^2/2 = \Delta E/\kappa\,, \tag{40}$$

where $\theta_R = \kappa t$.

5.2. KEPLERIAN APPROXIMATION

In case of the potential (Bičak et al. 1993):

$$\Phi = -\frac{GM}{b + \sqrt{R^2 + b^2}}\,, \tag{41}$$

we consider the simple case $b = 0$. The resulting approximation is exact for trajectories in the potential of a central point mass, corresponding to $b = 0$ (Kalnajs 1971). The relations for the radial motion are familiar from celestial mechanics:

$$R(t) = RE(1 - e\cos\eta)\,, \tag{42}$$

$$\theta_R = \kappa t = \eta - e\sin\eta\,, \tag{43}$$

$$e = \sqrt{\Delta L^2}\,/(R_E^2\kappa) = (\gamma/2)\sqrt{1 - L^2/L_\mathrm{C}^2}\,, \tag{44}$$

$$J_R = (2L_\mathrm{C}/\gamma)[1 - \sqrt{1 - e^2}\,]. \tag{45}$$

The result of these approximations is shown in Fig. 7.

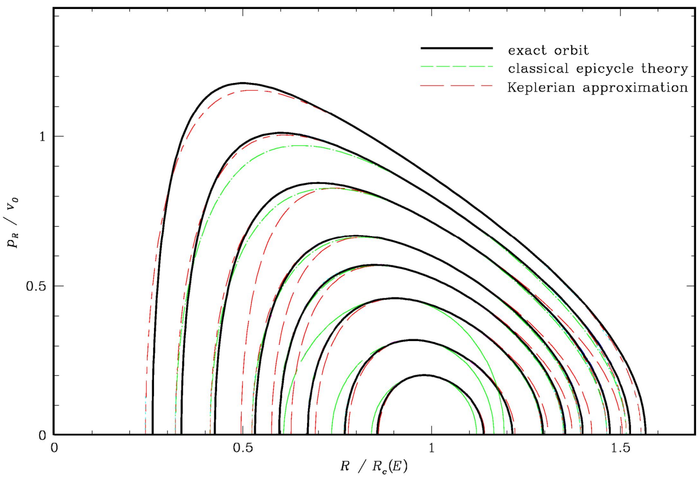

Fig. 7 – Surfaces of the R–p_R section of phase space: comparison of exact orbits in the logarithmic potential, which supports a flat rotation curve, with the predictions of the various orbit approximation. The orbital angular momenta are (from inside to outside) 0.973, 0.954, 0.902, 0.847, 0.796, 0.068, 0.597, and 0.490 of $L_C(E)$.

The classical epicycle theory of Lindblad is usually derived starting from the Hamiltonian or, equivalently, from the equations of motion for a near-circular orbit in a spherical or flat axisymmetric potential. Here, I directly considered integral (29) to be solved in order to obtain $R(t)$. The simplest way to approximate this integral by some closed form leads straightforwardly to the classical epicycle theory. I derive a general method for approximating this integral in a better way by manipulating the integrand.

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Received on 15 June 2010